Mathematical Reviews

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HISTORY

*Singer, Charles. **A Short History of Science to the Nineteenth Century.** Oxford University Press, New York, 1941. xiv+399 pp. $3.75.

In this well-written and well-illustrated book the author has set himself the difficult task of giving a condensed history of the development of science during all periods of history. The reviewer has no authority to judge the parts which deal with general sciences. As far as mathematics and astronomy are concerned, however, it is obvious that the discussion is based on totally antiquated material and viewpoints. Moreover, regarding a sentence (p. 84) like "there is no reason to suppose that the Almagest contains important original elements," it can only be said that the author could have found such reasons by opening this work and discovering, e.g., that the theory of the planets (books IX to XIII) is Ptolemy's own creation. Few works in human literature exercised a deeper influence on the history of science than this theory. It seems to me not unreasonable to expect historians to look occasionally into such works.

O. Neugebauer (Providence, R. I.).

Struik, D. J. On the sociology of mathematics. Science and Society 6, 58–70 (1942). [MF 6976]

*Selections Illustrating the History of Greek Mathematics. With an English translation by Ivor Thomas. II. From Aristarchus to Pappus.** The Loeb Classical Library. Harvard University Press, Cambridge, Mass., 1941. xi+683 pp. $2.50.

This is the concluding part of a selection of Greek mathematical works, the first part of which was reviewed in these Rev. 1, 33. Besides extracts from Archimedes, Apollonius and Diophantus, chapters are included concerning texts dealing with trigonometry, mensuration, the "Collection" of Pappus, etc. The two volumes together give a very fair impression of all types of the remains of Greek mathematics and constitute an excellent means of introduction to the study of the historical basis of mathematics.

O. Neugebauer (Providence, R. I.).

van der Waerden, B. L. Zur pythagoreischen Algebra: Quadratwurzel und Kubikwurzel. Math. Ann. 118, 286–288 (1941). [MF 6636]

van der Waerden, B. L. Die Astronomie der Pythagoreer und die Entstehung des geozentrischen Weltbildes. Himmelswelt 1941, 14 pp. (1941). [MF 6635]

The author gives an outline of the development of Greek astronomy in its earlier phases. He seems to have overlooked the book of E. Frank, Plato und die sogenannten Pythagoreer [Niemeyer, Halle, 1923], where essential points of his theory are already published. *O. Neugebauer.*

Neugebauer, O. Egyptian planetary texts. Trans. Amer. Philos. Soc. (N.S.) 32, 209–250 (1942). (27 plates) [MF 6247]

The paper contains editions of two Demotic texts, a large papyrus from Berlin and four wooden tablets from Liverpool. Both texts have been published previously [by Spiegelberg in 1902 and by Brugsch and Stobart in 1855–1856], but incompletely and with no or little commentary. An extensive discussion of the astronomical and historical problems involved is given. The astronomical commentary considers the relation to the actual movement of the planets during the times in question (from 16 B.C. to 132 A.D., with gaps) and establishes the date and calendar system of the two texts. Thereafter the author is concerned with the determination of the point in the ecliptic which was used for measuring longitudes. It is found that a fixed origin of about $\lambda = -4°$ was used. In the historical commentary the Greco-Roman tradition about the Egyptian planetary theories is discussed, and it is shown that they originated from a fictitious hypothesis as to the origin of Egyptian and Babylonian science. It is made very probable that these planetary tables are the "eternal tables" which are criticized by Ptolemy and which are also preserved in a Greek papyrus fragment from Tebtunis. This latter text is also re-edited here. At the end questions of terminology and paleography are discussed. The paper concludes with a bibliography and 27 plates. *O. Schmidt.*

Neugebauer, O. On some astronomical papyri and related problems of ancient geography. Trans. Amer. Philos. Soc. (N.S.) 32, 251–263 (1942). [MF 6248]

The texts discussed here are Greek papyri from the Michigan collection published in 1936. It is shown first that P.M. III 150 gives longitudes of the moon at opposition. The main part of this paper is devoted to texts which deal with the problem of the variation of the length of the day, both as a function of the seasons and of the geographical latitude [cf. the author's earlier paper, University of Pennsylvania Bicentennial Conference, Studies in the History of Science, Philadelphia, 1941, pp. 13–29; these Rev. 2, 306]. Here the "linear methods" are discussed in detail and it is shown that they are the generalization resulting from two types which are well known from Babylonian astronomy. This leads to the investigation of the concept of geographical "climate" in Greco-Roman literature. *O. Schmidt.*

von Erhardt, Rudolf and von Erhardt-Siebold, Erika. Archimedes' Sand-Reckoner. Aristarchos and Copernicus. Isis 33, 578–602 (1942). [MF 6361]

The authors discuss the famous passage in Archimedes' "Sand-Reckoner" where he reports about Aristarchus' heliocentric hypothesis. Aristarchus' theory must necessarily account for the absence of a yearly parallax of the fixed stars; this is contained in his sentence "that the circle, through which the earth revolves, has a ratio to the distance

of the fixed stars equal to that which in a sphere the center has to the surface." Archimedes comments "quite obviously this is impossible because the center of the sphere has no magnitude and therefore neither can it have a ratio to the surface of the sphere." The authors consider this criticism as "perfectly incomprehensible" [p. 587] and think that "Archimedes' mathematical equipment" should have made it not "difficult for him to grasp Aristarchus' ratio $x : dx = \infty$." They consider this argument a basis for doubting Archimedes' authorship. [See, however, the paper quoted below.]

It is made plausible that Copernicus could have known the "Sand-Reckoner." *O. Neugebauer.*

Neugebauer, O. Archimedes and Aristarchus. Isis **34**, 4–6 (1942). [MF 7043]

Woolard, Edgar W. Great astronomical treatises of the past. J. Washington Acad. Sci. **32**, 189–216 (1942). [MF 7037]

Woolard, Edgar W. The historical development of celestial co-ordinate systems. Publ. Astr. Soc. Pacific **54**, 14 pp. (1942). [MF 6519]

Sleight, E. R. Early English arithmetics. Nat. Math. Mag. **16**, 243–251 (1942). [MF 6261]

Schimank, Hans. Die Kunst-Rechnungs-liebende Societät als Gründung deutscher Schreib- und Rechenmeister. Mitt. Math. Ges. Hamburg **8**, 22–54 (1941). (1 plate) [MF 6939]

Boyer, Carl B. Cavalieri, limits and discarded infinitesimals. Scripta Math. **8**, 79–91 (1941). [MF 6480]

Rossier, Paul. Sur l'histoire de la rectification approximative d'un arc de cercle. C. R. Soc. Phys. Genève **58**, 170–171 (1941). [MF 6501]

Rossier, Paul. Lesage et la quadrature du cercle. C. R. Soc. Phys. Genève **58**, 204–207 (1941). [MF 6502]

Wisdom, J. O. The Analyst controversy: Berkeley's influence on the development of mathematics. Hermathena, no. **54**, 3–29 (1939). [MF 6766]

Wisdom, J. O. The compensation of errors in the method of fluxions. Hermathena, no. **57**, 49–81 (1941). [MF 6767]

Wisdom, J. O. The Analyst controversy: Berkeley as a mathematician. Hermathena, no. **59**, 111–128 (1942). [MF 6768]

The author reviews, and apparently shares, Berkeley's criticism of Newton's method of fluxions.

O. Neugebauer (Providence, R. I.).

***Euler, Leonhard. Commentationes Arithmeticae. V. 3.** Edited by Rudolf Fueter. Leonhardi Euleri Opera Omnia (1) 4. Orell Füssli, Zürich; B. G. Teubner, Leipzig, 1941. xxxiii+431 pp.

Brown, B. H. The Euler-Diderot anecdote. Amer. Math. Monthly **49**, 302–303 (1942). [MF 6692]

Somigliana, C. Questioni risolute e questioni da risolvere nella teoria del campo gravitazionale. Univ. e Politecnico Torino. Rend. Sem. Mat. Fis. **7**, 19–24 (1941). [MF 6278]

The author gives a brief sketch of the history of the theory of the earth's gravitational field, with particular reference to Clairaut's celebrated formula

$$g_p/c - g_e/a = 5w^2/2,$$

when g_p, g_e are the values of the gravitational accelerations at the pole and equator, respectively; a, c are the semi-axes of the terrestrial ellipsoid, and w is the angular velocity of rotation about the minor axis c. The modifications of this formula due to Stokes [Cambridge and Dublin Math. J. **4** (1849)] and Pizzetti [Atti R. Accad. Lincei. **3** (1864)] are pointed out; a resumé of the author's own contributions to this topic is also given. [For a brief summary of some of these results see Somigliana, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 597–611 (1940); these Rev. **3**, 116.]

M. A. Basoco (Lincoln, Neb.).

Cohen, I. Bernard. The first explanation of interference. Amer. J. Phys. **8**, 99–106 (1940). [MF 6515]

Ionescu, D. V. L'activité scientifique du Professeur A. Angelescu. Mathematica, Timişoara **17**, 111–128 (1941). [MF 6734]

Sergescu, P. Le Professeur G. Bratu. Mathematica, Timişoara **17**, 137–143 (1941). [MF 6736]

Baidaff, Bernardo I. Obituary: Claro Cornelio Dassen. Bol. Mat. **15**, 5–9 (1942). (Spanish) [MF 6601]

Chapman, S. and Whittaker, E. T. Obituary: Prof. A. R. Forsyth, F.R.S. Nature **150**, 49–50 (1942). [MF 7062]

Roth, L. Obituary: Prof. T. Levi-Civita, For. Mem. R. S. Nature **149**, 266 (1942). [MF 6391]

Obituary: Tullio Levi-Civita. Revista Ci., Lima **43**, 683–685 (1941). (Spanish) [MF 6642]

Mandelbrojt, S. Obituary: Emile Picard. 1856–1941. Amer. Math. Monthly **49**, 277–278 (1942). [MF 6688]

Stepanoff, W. Obituary: Ivan Ivanovitch Privaloff. 1891–1941. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 389–391; list of publications, 391–394 (1941). (Russian) [MF 6826]

Neville, E. H. Srinivasa Ramanujan. Nature **149**, 292–295 (1942). [MF 6457]

Whittaker, E. T. Obituary: Vito Volterra. 1860–1940. Obit. Notices Roy. Soc. London **3**, 691–718; list of publications, 718–729 (1941). (1 plate) [MF 6469]

Turnbull, H. W. Obituary: Alfred Young. 1873–1940. Obit. Notices Roy. Soc. London **3**, 761–777; list of publications, 777–778 (1941). (1 plate) [MF 6470]

ALGEBRA

Sergescu, P. Sur les combinaisons généralisées. Acad. Roum. Bull. Sect. Sci. **23**, 485–491 (1941). [MF 5551]

Miller, A. R. The number of configurations of a co-operative assembly. Proc. Cambridge Philos. Soc. **38**, 109–124 (1942). [MF 6038]

Freed from physical applications, the paper considers the combinatorial problem of finding an approximate expression for the number of ways in which N/n clusters of n points each can be placed on a lattice of N sites (that is, lattice-points), the essential requirement being that each cluster form a chain of points, consecutive pairs of which are one mesh-length apart. The cases studied are $n=2$ and 3, the lattices being rectangular or parallelepipedinal, or rectangular rolled into a cylinder. The methods are elementary but the explanations often insufficient for clarity.

B. O. Koopman (New York, N. Y.).

Allen, Edward S. and Diehl, Harvey. The number of stereoisomeric alcohols. Iowa State Coll. J. Sci. **16**, 161–171 (1942). [MF 5975]

To find the number of all the structurally different chemical compounds (aliphatic alcohols) contained in the formula $C_nH_{2n+1}OH$ is a curious problem of combinatorial analysis, attacked by Cayley in 1877 and considered more recently, along with some connected problems concerning stereoisomerids, by the chemists Henze and Blair and the reviewer. The authors carry the problem to further details, calculating the number of compounds in certain subclasses whose chemical characterization is of a certain interest but can not be given here. The authors work with recursive formulas, following the method of Henze and Blair, and distinguishing not less than 15 types of terms. They do not seem to know the related work of the reviewer. [The reviewer's $R_{n\alpha}$ defined in Acta Math. **68**, 145–254 (1937), see p. 195, and tabulated in Z. Kristallogr. Mineral. Petrogr. Abt. A. **93**, 415–443 (1936), see p. 439, is different from, though related to, the $T_{n\alpha}$ of the authors. In particular, $R_{n,2}=T_{n,2^n}$ for $n\leqq 12$, but this ceases to be generally true when $n\geqq 13$.] *G. Pólya* (Stanford University, Calif.).

Mirimanoff, D. Expressions de la somme de deux indéterminées en fonction du produit. Comment. Math. Helv. **14**, 1–22 (1942).

Let $f_r=\sum x_1\cdots x_r$ $(r=1,\cdots,n)$ be the elementary symmetric functions on the n quantities $x_1,\cdots,x_n$, and let $p=x_1+x_2$, $q=x_1x_2$. It is classical that each of p, q is a rational function of the other, with numerator and denominator having as coefficients polynomials in $f_1,\cdots,f_n$ with rational integral coefficients; and this fact has been used in an old proof of the fundamental theorem of algebra. However, computation of this rational relation by classical methods is long and complicated. The present paper develops an algorithm that simplifies both proof and effective construction. Let Q be the ring of polynomials in q, whose coefficients are polynomials in $f_1,\cdots,f_n$ with rational integral coefficients. The following results are proved: (i) For $n\geqq 4$, p has a "canonical" form $p=N(q)/D(q)$, with N, D in Q, where D is of degree $\frac{1}{2}n(n-3)$ in q and of degree $n(n-3)$ in the set $x_1,\cdots,x_n$; explicitly,

$$D=(x_1x_2)^{n-2}\cdot\prod_{i<j}(x_1x_2-x_ix_j),\quad i<j=3,\cdots,n.$$

(ii) For $n\geqq 4$, p has a "reduced" form $N_1(q)/D_1(q)$ with N_1, D_1 in Q, where D_1 is of degree $(n-2)^2$ in $x_1,\cdots,x_n$; and for every representation $p=N_2(q)/D_2(q)$, N_2, D_2 in Q, the degree of D_2 in $x_1,\cdots,x_n$ is at least $(n-2)^2$, so that the reduced form attains the minimum degree.

I. M. Sheffer (State College, Pa.).

Mirimanoff, D. Expressions de la somme x_1+x_2 de deux indéterminées x_1, x_2 en fonction de $x_1x_2+c(x_1+x_2)$. Comment. Math. Helv. **14**, 310–313 (1942).

Let $p=x_1+x_2$, $q=x_1x_2$ [see the preceding review]. It is sometimes necessary to express p as a rational function of $q+cp$, c being a suitably chosen constant, rather than of q alone, and this modification is carried out here in elementary fashion. *I. M. Sheffer* (State College, Pa.).

Linear Algebra

Oldenburger, Rufus. Matrix methods in the solution of algebraic equations. Amer. Math. Monthly **49**, 310–315 (1942). [MF 6694]

Let $f(x)=a_0x^n+na_1x^{n-1}+\cdots+a_n$ be a polynomial and define

$$A_i=\begin{Vmatrix} a_0 & a_1 & \cdots & a_i \\ a_1 & a_2 & \cdots & a_{i+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-i} & a_{n-i+1} & \cdots & a_n \end{Vmatrix},\quad i=0,1,\cdots,n.$$

Let m be the smallest value of i such that there exists an i-ic form F_i in $(x, 1)$ apolar to each of the forms whose coefficients are the rows of A_i. Then $f(x)=0$ has an r-fold root α, $r\geqq m$, if and only if α is an m-fold root of $F_m(x)=0$.

C. C. MacDuffee (New York, N. Y.).

Oldenburger, Rufus. Repeated linear factors of forms. Amer. J. Math. **64**, 592–596 (1942). [MF 6951]

Let $\mathfrak{F}$ be a completely symmetric algebraic form of degree n in m variables with the n-way matrix $\mathfrak{A}$. The elements of $\mathfrak{A}$ can be arranged to form a 2-way matrix $\mathfrak{A}_\sigma$ $(\sigma=0,1,\cdots,n)$ by choosing σ indices of $\mathfrak{A}$ for the multipartite column index of $\mathfrak{A}_\sigma$, and $n-\sigma$ indices of $\mathfrak{A}$ for the row index of $\mathfrak{A}_\sigma$. It is shown that $\mathfrak{F}$ has a linear factor $\mathfrak{L}$ repeated r times if and only if $\mathfrak{L}^{n-r+1}$ is a certain covariant constructible from the matrix $\mathfrak{A}_{n-r+1}$. *C. C. MacDuffee* (New York, N. Y.).

Guareschi, Giacinto. Alcune identità tra matrici. Atti Accad. Ligure **1**, 233–238 (1941). [MF 6398]

The object of this paper is to give a proof of the identity below, which the author declares arises in the study of intermediate ("intermedie") curvatures of a curve in hyperspace. Contrary to the author's assertion it is an identity of determinants rather than matrices. Let A be the matrix (a_{ij}) with m rows and n columns. Define

$$U=\begin{pmatrix}A\\u\end{pmatrix},\quad V=\begin{pmatrix}A\\v\end{pmatrix},\quad W=\begin{pmatrix}A\\u\\v\end{pmatrix},$$

where u and v are row matrices of n elements. The identity is:

$$|UU'|\cdot|VV'|-|UV'|^2=|AA'|\cdot|WW'|,$$

where the primes denote "transpose" and $|X|$ is the determinant of X. *B. W. Jones* (Ithaca, N. Y.).

Schwerdtfeger, H. On generalized Hermitian matrices. Amer. Math. Monthly **49**, 181–184 (1942). [MF 6395]

A matrix over a field is defined to be a P_r-matrix if it is of rank r and if there exist r linearly independent rows such that the corresponding columns (that is, columns with the same indices) are also linearly independent. The author shows that in every P_r-matrix there exists an r-rowed principal submatrix of rank r. He uses this result in a short discussion of ϕ-symmetric matrices. [Cf. A. A. Albert, Modern Higher Algebra, University of Chicago Press, Chicago, Ill., 1937, Chap. V.] *J. Williamson.*

Bose, S. N. Studies in Lorentz group. Bull. Calcutta Math. Soc. **31**, 137–147 (1939). [MF 6148]

The author proves that any proper 4×4 orthogonal matrix whose elements are complex numbers can be written as a product of two commutative matrices, each corresponding to a spinor transformation of a special type. The result is well known in spinor theory; this proof, which is quite elementary in nature, uses a suitably chosen set of 16 linearly independent 4×4 matrices as a linear basis for all 4×4 matrices. Several of the equations have typographical errors, and the author's Proposition II is not carefully stated. *A. Schwartz* (State College, Pa.).

von Neumann, John. Approximative properties of matrices of high finite order. Portugaliae Math. **3**, 1–62 (1942). [MF 6235]

Let M_n denote the set of matrices A on an n-dimensional space E_n. Let $t(A)$ denote the "average trace," that is, $t(A)=(1/n)\,\mathrm{Tr}\,(A)$ and $\|A\|^2=t(A^*A)$; $\|A\|^2$ is also the inner product of Af with itself if f is the vector $\{\alpha, \cdots, \alpha\}$, where $\alpha=1/n^{\frac{1}{2}}$. This paper is concerned with the characteristics of the set M_n relative to the norm $\|A\|$. If m is a factor of n, M_m is isomorphic with a subset of M_n and this isomorphism preserves the norm. This justifies the use of $t(..)$ rather than $\mathrm{Tr}(..)$ in defining $\|A\|^2$. An element B of M_n which is the image of an element of M_m under such an isomorphism can be termed essentially m-dimensional. In the present paper it is shown that there is an $\epsilon>0$ such that for all m and $n=pm$ there is a matrix A in M_n with bound $|||A|||\leqq 1$ for which there is no essentially m-dimensional B such that $\|A-B\|<\epsilon$.

In the proof, one first establishes the corresponding result in which the condition "$|||A|||\leqq 1$" is replaced by "$\|A\|\leqq 1$." If B is essentially m-dimensional and k is an integer less than n/m, then there is a projection E with $\|E\|^2=km/n$ such that $BE=EB$. Thus one considers the set $S_{k,\epsilon}$ of matrices $A\epsilon M_n$ with $\|A\|\leqq 1$ such that there is a B with the properties (a) there is a projection E for which $BE=EB$ and $\delta<\|E\|<1-\delta$ and (b) $\|A-B\|<\epsilon$. If, in this definition, we replace the condition $\|A\|\leqq 1$ by the condition $|||A|||\leqq 1$, we will denote the resulting set by $\bar{S}_{k,\epsilon}$. It is shown that for each $\delta<\frac{1}{2}$ there is an ϵ such that for all n the number of spheres of a certain radius needed to cover $S_{k,\epsilon}$ is smaller than the number of spheres of the same radius needed to cover $M_{n,1}$, the subset of M_n with $\|A\|<1$. This "volumetric" argument shows that $S_{k,\epsilon}$ is a proper subset of $M_{n,1}$ and hence, a fortiori, the subset of $S_{k,\epsilon}$ consisting of those A's for which there is an essentially m-dimensional B with $\|A-B\|<\epsilon$ is a proper subset of $M_{n,1}$. If $0<\delta<\frac{1}{4}$, this last is the desired result, at least for the condition $\|A\|\leqq 1$. The discussion is based on the fact that, if B commutes with E, there is a unitary U such that $B=UB'U^*$, $E=UE'U^*$, where B' has only zeros above the main diagonal and E' has a diagonal matrix. The spatial arrangement of the unitary transforms is obtained by expressing an arbitrary U in terms of a Hermitian A with $|||A|||\leqq 1$, $U=(1+iA)^2/(1-iA)^2$ and deriving certain inequalities.

Starting with an A which is not in $S_{k,\epsilon}$, we obtain one that is not in $\bar{S}_{k,\epsilon}$ by a process called "cutting through a thin part of the spectrum." By a logarithmic division of the spectral interval of A^*A and AA^*, it is shown that certain segments make a relatively light contribution to A and A^*. These intervals and the part above are cut off by taking FAF for a suitable projection F and the result normalized to an $A'=kFAF$ with bound 1. By this process the part cut off will have dimensionality $n\eta$, where η is small relative to $1/k$. (A straightforward procedure would yield the ineffective result $\eta\leqq 1/k$.) Hence A' will not be in $\bar{S}_{k,\epsilon}$ for a slightly different δ and ϵ.

Let $\bar{T}_{k,\epsilon}$ denote the set of A's with $|||A|||\leqq 1$ and such that there is a projection E with $\delta<\|E\|<1-\delta$ for which $\|AE-EA\|<\epsilon$ and $\bar{R}_{k,\epsilon}$ denote the set of A's with $|||A|||\leqq 1$ and such that there is a symmetric B with $\|B-t(B)\cdot 1\|>\delta$ such that $\|AB-BA\|<\epsilon$. It is also shown that to each $\delta<\frac{1}{2}$ there is an $\epsilon>0$ such that $\bar{T}_{k,\epsilon}$ is not the full set of A's with $|||A|||\leqq 1$, and the corresponding result for $\bar{R}_{k,\epsilon}$. In the first appendix the author discusses the imbedding of an M_m in an M_n when $n=pm$. In the second appendix it is shown that a normal A of M_n with $|||A|||\leqq 1$ can be approximated by essentially m-dimensional elements $B(n=pm)$ to within ϵ, if m is sufficiently large. This is, of course, the opposite of the situation for a general A. In the third appendix, the formula

$$\|AB-BA\|^2-\|AB^*-B^*A\|^2=t((A^*A-AA^*)(B^*B-BB^*))$$

is established. This formula has the interesting consequence that, if $AB=BA$ and either A or B is normal, then $AB^*=B^*A$. *F. J. Murray* (New York, N. Y.).

McCoy, Neal H. Algebraic properties of certain matrices over a ring. Duke Math. J. **9**, 322–340 (1942). [MF 6871]

This paper is an abstraction and extension of work of E. H. Moore [Moore-Barnard, General Analysis, Pt. I, American Philosophical Society, Philadelphia, 1935] on Hermitian matrices. Let $\mathfrak{R}$ be an arbitrary ring with unit element, $\mathfrak{R}_n$ the class of all matrices of order n over $\mathfrak{R}$, and $\mathfrak{R}_n'$ a subclass of $\mathfrak{R}_n$ consisting of those matrices whose elements satisfy certain weakened commutativity conditions which are sufficient for a satisfactory definition of determinant to exist. It is first established that if $A\epsilon\mathfrak{R}_n'$, then adj $A\epsilon\mathfrak{R}_n'$, and adj A can be written as a polynomial in A with coefficients in the center $\mathfrak{C}$ of $\mathfrak{R}$. If $A\epsilon\mathfrak{R}_n'$ and A is a divisor of zero in $\mathfrak{R}_n$, then $|A|$ is a divisor of zero in $\mathfrak{R}$; if A has an inverse in $\mathfrak{R}_n$ and $|A|$ is not a divisor of zero in $\mathfrak{R}$, then $|A|$ has an inverse in $\mathfrak{R}$. The totality of all elements $h(\lambda)$ of $\mathfrak{C}[\lambda]$, such that $h(A)=0$, is an ideal $\mathfrak{m}$ in $\mathfrak{C}[\lambda]$ called the minimum ideal. If $f(\lambda)$ is the characteristic polynomial of A, $(f(\lambda))$ and $\mathfrak{m}$ have the same prime ideal divisors in $\mathfrak{C}[\lambda]$. A theorem of Ostrowski [Quart. J. Math., Oxford Ser. **9**, 241–245 (1938)] is extended to sets of elements of $\mathfrak{R}_n'$. Some further theorems are obtained in the special case where $\mathfrak{R}$ is commutative, or $\mathfrak{R}$ is a quaternion ring and $\mathfrak{R}_n'$ consists of Hermitian matrices.
C. C. MacDuffee (New York, N. Y.).

Jones, Burton W. An extension of a theorem of Witt. Bull. Amer. Math. Soc. **48**, 133–142 (1942). [MF 6193]

If $u_1, \cdots, u_n$ is a set of vectors such that $u_iu_j=u_ju_i=a_{ij}$ $(i, j=1, 2, \cdots, n)$, where the a_{ij} are elements of a field K,

then all linear combinations of these vectors with coefficients in K constitute a vector space $\mathfrak{S}=\langle u_1, \cdots, u_n\rangle$ over K. We assume that $|a_{ij}|\neq 0$, and make a like assumption for all spaces considered. The vector space $\mathfrak{S}$ may be made into a metric space by defining the inner product of two vectors $\sum x_i u_i$ and $\sum y_i u_i$ to be $\sum a_{ij}x_i y_j$. If $\mathfrak{C}$ is a nonsingular matrix, with coefficients in K, such that $(u_1, \cdots, u_n)\mathfrak{C}=(v_1, \cdots, v_n)$, then the space $\mathfrak{S}'$ defined by $v_1, \cdots, v_n$ is equivalent to $\mathfrak{S}$; in symbols, $\mathfrak{S}\cong\mathfrak{S}'$. The notation $\mathfrak{S}=\mathfrak{S}_1+\mathfrak{S}_2$ is used to indicate that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are complementary orthogonal subspaces of $\mathfrak{S}$.

E. Witt [J. Reine Angew. Math. 176, 31–44 (1936)] has shown that, if $\mathfrak{S}_1$, $\mathfrak{S}_2$ and $\mathfrak{S}_3$ are vector spaces over a field K of characteristic other than 2 and $\mathfrak{S}_2$ and $\mathfrak{S}_3$ are orthogonal to $\mathfrak{S}_1$, then $\mathfrak{S}_1+\mathfrak{S}_2\cong\mathfrak{S}_1+\mathfrak{S}_3$ implies that $\mathfrak{S}_2\cong\mathfrak{S}_3$. The present paper shows that this result holds if the field K is replaced throughout by any ring of p-adic integers with p an odd prime. Witt's result, as well as this extension, may also be expressed in the language of equivalence of quadratic forms. *N. H. McCoy* (Northampton, Mass.).

Albert, A. A. Quadratic forms permitting composition. Ann. of Math. (2) **43**, 161–177 (1942). [MF 6068]

Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be independent indeterminates over a field $\mathfrak{F}$. A quadratic form $f(x)$ in $x_1, \cdots, x_n$, with coefficients in $\mathfrak{F}$, is said to permit composition if $f(x)f(y)=f(z)$, where z_k is a properly chosen bilinear form in the two sets of indeterminates $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ $(k=1, 2, \cdots, n)$. An alternative algebra over $\mathfrak{F}$ is an algebra such that $x(xy)=(xx)y$ and $(yx)x=y(xx)$, for every x and y of the algebra. [See Zorn, Abh. Math. Sem. Hansischen Univ. **8**, 123–147 (1930); **9**, 395–402 (1933).] The interest here centers in alternative algebras with a unity quantity and an involution J such that $x+x^J$ and xx^J are in $\mathfrak{F}$. The norm form xx^J of such an algebra is a quadratic form in the coordinates of x which permits composition and, conversely, a quadratic form over $\mathfrak{F}$ which permits composition is equivalent in $\mathfrak{F}$ to the norm form of such an algebra. These norm forms are quadratic forms in 1, 2, 4 or 8 indeterminates except for the diagonal norm forms, in 2^t indeterminates, of purely inseparable fields of degree 2^t and exponent two over $\mathfrak{F}$ of characteristic two. For the case in which $\mathfrak{F}$ is algebraically closed and of characteristic not two, these results are due to Hurwitz [Nachr. Ges. Wiss. Göttingen **1898**, 309–316] and Dickson [Ann. of Math. (2) **20**, 155–171 (1919)]. In this paper, the author presents a unified study for arbitrary $\mathfrak{F}$. One feature is the introduction of some new algebras of order 2^e over $\mathfrak{F}$ which generalize the Cayley-Dickson algebras of order eight.

N. H. McCoy (Northampton, Mass.).

Dines, Lloyd L. On the mapping of n quadratic forms. Bull. Amer. Math. Soc. **48**, 467–471 (1942). [MF 6715]

Let $Q_1(z), Q_2(z), \cdots, Q_n(z)$ be real quadratic forms in the real variables $z^1, z^2, \cdots, z^m$. A number of writers have sought conditions under which there exists a real linear combination of these forms which is positive definite. For references, see a previous paper by the present author [Bull. Amer. Math. Soc. **47**, 494–498 (1941); these Rev. **2**, 341] which is devoted to the case $n=2$. The given forms may be used to define a map $\mathfrak{M}$ of the m-dimensional space $\mathfrak{Z}_m$ of points $z=(z^1, z^2, \cdots, z^m)$ onto an n-dimensional space $\mathfrak{X}_n$ of points $x=(x^1, x^2, \cdots, x^n)$, where $x=(Q_1(z), Q_2(z), \cdots, Q_n(z))$, z on $\mathfrak{Z}_m$. For $n=2$ the map $\mathfrak{M}$ is convex, but this is no longer true for $n>2$. However, by making use of the convex extension of a suitably chosen subset of $\mathfrak{M}$, the author is able to establish the following generalization of his earlier result: A necessary and sufficient condition that there exist a linear combination $\sum_{i=1}^{n}\lambda_i Q_i(z)$ such that $\sum_{i=1}^{n}\lambda_i Q_i(z)>0$ for $z\neq 0$ is that there exist in $\mathfrak{Z}_m$ no set of points $z_j\neq 0$ $(j=1, 2, \cdots, r)$ such that $\sum_{j=1}^{r}m_j Q_i(z_j)=0$ $(i=1, 2, \cdots, n)$, the coefficients m_j being positive. A similar condition is also given for the existence of a semidefinite linear combination of the given forms. *N. H. McCoy.*

Abstract Algebra

Klein-Barmen, Fritz. Über Verbände mit einer weiteren assoziativen und kommutativen Elementverknüpfung. Math. Z. **47**, 85–104 (1940). [MF 6300]

The author considers a lattice over which an additional commutative and associative operation of multiplication is defined and satisfying one or more of the following postulates: KI: $(a\cup b)(a\cap b)=ab$; KII′: $a(b\cup c)=ab\cup ac$; KII″: $a(b\cap c)=ab\cap ac$; AIV: $ab=ab'$ implies $b=b'$. Various elementary relations between these postulates are proved and under the assumption of KI and the distributive law a generalization of KI for n elements is shown to hold.

R. P. Dilworth (New Haven, Conn.).

Whitman, Philip M. Free lattices. II. Ann. of Math. (2) **43**, 104–115 (1942). [MF 6064]

This is a continuation of the author's researches into the fundamental properties of the free lattice $FL(n)$ with n generators [Ann. of Math. (2) **42**, 325–330 (1941); these Rev. **2**, 244]. He now proves (i) the group of automorphisms of $FL(n)$ is the symmetric group on n symbols; (ii) $FL(3)$ contains $FL(n)$ as a sublattice, for any finite or countable n; (iii) the lattice operations in $FL(n)$ are continuous (possibly vacuously) in the intrinsic order topology; (iv) $FL(n)$ is not complete. The paper concludes with a proof [of the reviewer] that the free modular lattice with four generators contains an infinite chain, and a list of five interesting unsolved problems. *G. Birkhoff.*

Ward, Morgan. The closure operators of a lattice. Ann. of Math. (2) **43**, 191–196 (1942). [MF 6458]

The author treats certain abstract imbedding problems by means of closure operators on a lattice $\mathfrak{S}$: these are the single-valued functions φ for which $A\supset B$ implies $\varphi(A)\supset\varphi(B)$, $\varphi(A)\supset A$ for every A, $\varphi(\varphi(A))=\varphi(A)$ [G. Birkhoff, Lattice Theory, American Mathematical Society, New York, 1940, p. 17]. It is shown that the set of values of a closure operator, that is, the "closed" elements, forms a complete lattice in $\mathfrak{S}$. Conversely a subset $\mathfrak{S}'$ of $\mathfrak{S}$ containing the unit of $\mathfrak{S}$, and closed under cross-cut, determines a closure operator φ whose set of closed elements is $\mathfrak{S}'$. Further, given an arbitrary subset $\mathfrak{T}$ containing the unit of $\mathfrak{S}$, one obtains a lattice $\mathfrak{S}'$ containing $\mathfrak{T}$ by adjoining the cross-cuts of all sets of its elements; the closure operator belonging to $\mathfrak{S}'$ is called the imbedding operator of $\mathfrak{T}$. The "Dedekind cuts" of MacNeille are reexamined in terms of such imbedding operators. [Reference is made several times in the text to a paper cited as Ward-Dilworth 2, although no such paper is listed in the bibliography; presumably the reference is to Ann. of Math. (2) **40**, 600–608 (1939); these Rev. **1**, 2.] *H. Wallman.*

Gorn, Saul. Homomorphisms and modular functionals. Trans. Amer. Math. Soc. **51**, 103–116 (1942). [MF 6090]

This paper deals with a complemented modular lattice L with 0 and I. Denoting by C the set of complements the

author calls an ideal $\mathfrak{a}$ "neutral" (following Garrett Birkhoff) if $CC\mathfrak{a}=\mathfrak{a}$; a pair of neutral ideals $\mathfrak{a}$ and $\mathfrak{b}$, one multiplicative and the other additive, such that $\mathfrak{a}=C\mathfrak{b}$, is called a "$C$-neutral pair." It is shown that every C-neutral pair determines a congruence and that all homomorphisms arise from such congruences; this generalizes the situation prevailing in Boolean algebras. "Normal ideals," which are the same thing as MacNeille's cuts, are then defined and shown to provide the proper extension of Stone's "normality" in Boolean algebras. The author introduces "ideal irreducibility," stronger than irreducibility if L is not complete, and proves that $L/\mathfrak{a}$ is ideally irreducible if and only if $\mathfrak{a}$ is maximal. Finally it is shown that a norm satisfying the conditions of Wilcox and Smiley is unique if and only if the lattice is ideally irreducible. *H. Wallman.*

Pankajam, S. Ideal theory in Boolean algebra and its application to deductive systems. Proc. Indian Acad. Sci., Sect. A. **14**, 670–684 (1941). [MF 6479]

The ideals and dual ideals of a distributive lattice L are studied, with special emphasis on the case that L is a Boolean ring. The emphasis is on exposition of ideas of Stone and Tarski. It is recalled that the ideals of a Boolean ring A form a complete distributive lattice B in which joins are completely distributive. The Stone-Tarski theory of ideals (normal ideals, simple ideals, etc.) holds abstractly in any such B. The author introduces as a new concept the "Boolean complement" of an ideal as the dual ideal of the complements of its members. *G. Birkhoff.*

Everett, C. J. Vector spaces over rings. Bull. Amer. Math. Soc. **48**, 312–316 (1942). [MF 6412]

Let $M=u_1K+\cdots+u_mK$ be a vector space over a ring K, let N be a submodul of M and let $b(N)$ denote the basis number of N (when it exists). This paper is concerned with properties of the function $b(N)$ for various types of rings K. Typical theorems are: (C) $b(M)$ exists and is equal to m if the right ideals of K satisfy the ascending chain condition. (D) There exists a ring K over which any M_m is always an M_1. (F) $b(N)$ exists and is not greater than m for every submodul N of M if and only if every right ideal of K is principal of type rK, where r is a non-left-zero-divisor; moreover (G) $b(N)$ is a positive modular functional. (I) The condition $N<M$ implies $b(N)<m$ (that is, $b(N)$ is sharply positive) if and only if K is a quasi-field.

C. C. Torrance (Cleveland, Ohio).

Rosenbloom, Paul C. Post algebras. I. Postulates and general theory. Amer. J. Math. **64**, 167–188 (1942). [MF 5996]

By a Post algebra the author means an abstract generalization of the n-valued logic of E. L. Post, in the same sense in which a Boolean algebra is an abstract generalization of the two-valued logic. He gives a system of eleven postulates for Post algebras in terms of the operations $p+q$ and p', corresponding to Post's sum V_n and cyclic negation $\sim_n$. All the other operations of Post can be defined in terms of these two. Roughly speaking, these postulates require the algebra to be a distributive lattice (with zero and unit element) with respect to the addition operation $p+q$ taken as lattice join, and having a cyclic negation operation p' of period n, whose relation to the addition operation is necessarily quite complicated. From these postulates the general theory of Post algebras is derived in 68 theorems, the most important of which is the fundamental expansion theorem T53. This states that, if $f(p)$ is a function definable in terms of addition and negation, then $f(p)=\sum f(i)p_{-i}$, where i runs from 0 to $n-1$, $f(i)$ depends only on f and p_{-i} depends only on p. There is a similar expansion theorem for functions of several variables. The remaining 28 theorems of the paper deal with finite Post algebras. After defining prime elements and powers of primes, and proving a unique factorization theorem, the author shows that, if K is an n-valued Post algebra with m elements, then $m=n^k$, and K is isomorphic to the direct sum of k identical n-valued Post logics. It follows that two n-valued Post algebras with the same number of elements are isomorphic. *O. Frink.*

Thrall, R. M. On symmetrized Kronecker powers and the structure of the free Lie ring. Amer. J. Math. **64**, 371–388 (1942). [MF 6439]

Let K be a field of characteristic zero and consider the free distributive nonassociative K-ring Z_n with n generators $z_1, z_2, \cdots, z_n$. Any nonsingular linear transformation A of the z_i produces an automorphism of Z_n; in particular, the set Z_n^r consisting of all forms of degree r in the z_i undergoes a linear transformation $A[r]$. The mapping $A\rightarrow A[r]$ defines a representation of the full linear group $GL(n, K)$. If $\mathbf{A}$ is a given subgroup of $GL(n, K)$, and if J is a homogeneous ideal in Z_n which is invariant under every A in $\mathbf{A}$, then we obtain a new representation space of $\mathbf{A}$ by replacing every element of Z_n^r by its residue class (mod J). Representations of this type are called representations of power type of $\mathbf{A}$. Examples are the Kronecker rth power $A\times A\times\cdots\times A$, and the symmetrized Kronecker rth power $S_r(A)$. A new representation of this type is the rth "Lie representation"; here J is the ideal generated by the elements aa and $a(bc)+b(ca)+c(ab)$, $(a, b, c$ in $Z_n)$. A general method is developed for decomposing an integral rational representation of $GL(n, K)$ into irreducible constituents. In particular, the analysis of the rth symmetrized Kronecker power of the mth symmetrized Kronecker power is given in the cases (1) m arbitrary, $r\leqq 3$; (2) $m=2$, r arbitrary. A recursion formula is obtained for the decomposition of the rth Lie representation into irreducible representations, and the explicit expressions are computed for $r\leqq 10$.

R. Brauer (Toronto, Ont.).

McCoy, N. H. Remarks on divisors of zero. Amer. Math. Monthly **49**, 286–295 (1942). [MF 6690]

Let a be an element of a commutative ring R with unit element. An element $b\neq 0$ such that $ab=0$ is an annihilator of a. A matrix M is of rank r if the determinant of every square minor of M of order $r+1$ is annihilated by one and the same element of R, and if this is not true for minors of order r. A system of linear homogeneous equations has a nontrivial solution if and only if the rank of the coefficients is less than the number of unknowns. If $f(\lambda, \mu, \cdots)$ is a divisor of zero in the domain $R[\lambda, \mu, \cdots]$, it is annihilated by a number of R. The paper closes with an exposition of the imbedding theorems for rings with particular attention to nilpotent rings. *C. C. MacDuffee* (New York, N. Y.).

Tihomirov, A. Eine Verallgemeinerung des Begriffes des verschränkten Produktes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 297–304 (1941). (Russian. German summary) [MF 5837]

The crossed product proposed here is more general than the crossed product leading from a field and a group of its automorphisms by means of a factor set to an algebra; it is more general first in that a not necessarily finite semi-group (a set closed under multiplication) of automorphisms is used

instead of a finite group, and second in that the "factor set" is not assumed to satisfy the condition that leads to associativity. For such crossed products which thus result in (not necessarily associative) rings seven theorems are proved; some of them give conditions for the existence of a unit element and a centrum, others give structural properties which appear as generalizations of known properties of less general crossed products, including the statements that the algebra which is such a product is normal and simple.

G. Y. Rainich (Ann Arbor, Mich.).

MacLane, Saunders and Schilling, O. F. G. A formula for the direct product of crossed product algebras. Bull. Amer. Math. Soc. **48**, 108–114 (1942). [MF 6187]

The authors prove the following extension of the well-known formula for crossed products: $(N, \Gamma, h)\times(K, \Gamma/\Delta, G) \sim (N, \Gamma, hg)$. Here K is a normal subfield of the normal field N, Γ is the Galois group of N over the base field F, Δ is the subgroup belonging to K, h and G are factor sets and g is the factor set obtained from G by setting $g_{S,T}=G_{\sigma,\tau}$ for S in the coset σ of Γ/Δ and T is in the coset τ. The proof is essentially the same as for the special case $K=N$. On the basis of this theorem a number of theorems on crossed products are deduced in a very simple manner. The most comprehensive of these is a formula for the direct product of two crossed products whose defining normal fields have an arbitrary intersection. *N. Jacobson.*

Schmidt, F. K. and MacLane, Saunders. The generation of inseparable fields. Proc. Nat. Acad. Sci. U. S. A. **27**, 583–587 (1941). [MF 5909]

Let K be an extension of a field k of characteristic p such that K preserves p-independence over k. Then it is shown that if the transcendence degree of K is denumerable or finite then K is a limit of a tower of fields $k\subset L\subset M_0\subset M_1\subset\cdots$, where (i) K can be generated by L and by the pth powers from K, (ii) L has a separating transcendence basis over k, (iii) K/L preserves p-independence and L is algebraically closed in K, (iv) M_n is algebraic over M_{n-1}, (v) M_n has a separating transcendence basis over L and (vi) M_n can be generated from L by the adjunction of pth powers of elements that are in M_{n+1} but not in M_n. This theorem and its extension to fields of arbitrary transcendence degree may be applied to prove the known theorem that a p-adic field $\mathfrak{K}$ is uniquely determined by its field of residue classes. A number of conditions that an extension K of k preserve p-independence are obtained. *N. Jacobson.*

Lubelski, S. Verallgemeinerung und Umkehrung eines Hilbertschen Satzes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 399–400 (1941). (Russian. German summary) [MF 6828]

The author proves that a necessary and sufficient condition for the existence of a normal basis of an absolutely Abelian field K is that the discriminant of every cyclic subfield of K whose degree is a power of a prime be relatively prime with that degree. The author considers this as a generalization of a theorem of Hilbert of 1897. More recent literature is not taken into consideration.

G. Y. Rainich (Ann Arbor, Mich.).

Behrens, Ernst-August. Über die Existenz von Algebren beliebigen Ranges mit quadratischer Normenform. Math. Ann. **118**, 85–93 (1941). [MF 6311]

The results of Hecke [Math. Ann. **114**, 1–28, 316–351 (1937)] on theta series formed with quadratic forms might lead one to the conjecture that these quadratic forms are connected with the norms of certain linear algebras. In order to evaluate this conjecture the author considers the possible existence of algebras having the desired kind of norms. The results are negative. If the number of basis elements is different from 1, 2 and 4, the author shows that the usual methods will not yield norms which are quadratic forms or powers of quadratic forms, and that no quadratic form will give a norm which is multiplicative. The proofs are carried out by showing that if such a norm existed then the algebra would not be associative. *H. S. Zuckerman.*

Scott, Winston M. On matrix algebras over an algebraically closed field. Ann. of Math. (2) **43**, 147–160 (1942). [MF 6067]

Let K be an algebraically closed field; let $\mathfrak{A}$ be an algebra with unit element over K. If $\mathfrak{N}$ is the radical of $\mathfrak{A}$, then $\mathfrak{A}$ can be written as a direct sum of $\mathfrak{N}$ and a semisimple subring $\mathfrak{A}^*$, that is, $\mathfrak{A}=\mathfrak{A}^*+\mathfrak{N}$. The author considers (1-1)-representations $\mathfrak{C}$ of $\mathfrak{A}$; assume that the coordinate system is chosen so that $\mathfrak{C}$ breaks up into irreducible parts $\mathfrak{C}=(\mathfrak{C}_{ij})$ with $\mathfrak{C}_{ij}=0$ for $i<j$, and that the induced representation of $\mathfrak{A}^*$ breaks up completely, that is, $\mathfrak{C}_{ij}(A^*)=0$ for A^* in $\mathfrak{A}^*$ and $i>j$. The set of matrices $\mathfrak{C}_{ij}(A)$, with fixed i, j, can be considered as an $(\mathfrak{A}, \mathfrak{A})$-module, that is, a module with $\mathfrak{A}$ both as left and right operator system, multiplication being defined by $B\circ\mathfrak{C}_{ij}(A)=\mathfrak{C}_{ij}(B^*A)$, $\mathfrak{C}_{ij}(A)\circ B=\mathfrak{C}_{ij}(AB^*)$, where A and B are elements of $\mathfrak{A}$ and where B^* is the element of $\mathfrak{A}^*$ corresponding to B. These $\mathfrak{C}_{ij}$ are termed the simple parts of $\mathfrak{C}$. It is then shown that each $\mathfrak{C}_{ij}$ is either 0 or consists of all matrices with a given number of rows and columns. The $\mathfrak{C}_{ij}\neq 0$ are isomorphic to the factor groups of a composition series of $\mathfrak{A}$, where $\mathfrak{A}$ itself is considered as an $(\mathfrak{A}, \mathfrak{A})$-module. Conversely, to every factor group there corresponds at least one $\mathfrak{C}_{ij}$. For each factor group such a $\mathfrak{C}_{ij}$, an "elementary module" of $\mathfrak{A}$, is selected in a certain manner. It is then shown that every representation can be replaced by a similar representation such that all simple parts are linear combinations of the elementary modules of $\mathfrak{A}$. Finally, the decomposition of $\mathfrak{A}$ into a direct sum of ideals and the corresponding "block" distribution of the irreducible constituents is studied. *R. Brauer.*

Wade, L. I. Certain quantities transcendental over $GF(p^n, x)$. Duke Math. J. **8**, 701–720 (1941). [MF 5954]

Carlitz [Duke Math. J. **1**, 137–168 (1935)] has introduced several important quantities into the analysis of the field which is constituted by the power series in x^{-1/q^k} with coefficients from a finite field $GF(p^n)=GF(q)$. Let

$$[k]=x^{q^k}-x, \qquad L_k=[k][k-1]\cdots[1],$$
$$F_k=[k][k-1]^q\cdots[1]^{q^k}, \qquad \xi=\lim [1]^{q^k/(q-1)}/L_k,$$
$$\psi(t)=\sum(-1)^k t^{q^k}/F_k.$$

The function $\psi(t)$ shares with the "linear" polynomials $f=\sum A_i t^{q^i}$ the property $\psi(u+v)=\psi(u)+\psi(v)$; it has a multiplication theorem of the form $\psi(Mt)=f_M(\psi(t))$, where M is an "integer," that is, a polynomial form in x. Furthermore, $\psi(\xi)=0$ and $\psi(t+M\xi)=\psi(t)$.

The principal result of the present paper is the theorem that the values of the ψ-function (which is akin to the exponential function) are transcendental if the argument is an algebraic "number" not equal to 0. More specifically, it is shown: (1) Let $\zeta=\psi(1)=\sum(-1)^k/F_k$, or even $\sum B_k/F_k$ with certain restrictions on the degrees of the B_k, or finally $\zeta=\sum 1/[k]$. If f is a "linear" polynomial not equal to 0, then $f(\zeta)\neq 0$. (2) Let γ_i be the conjugates of an "algebraic

integer" γ; then the product of the values $\psi(\gamma_i)$ and a fortiori $\psi(\gamma)$ do not vanish. (3) Under the same conditions for the γ_i, with f a "linear" polynomial whose coefficients are only slightly restricted, the product of the quantities $f(\psi(\gamma_i))$ and a fortiori $f(\psi(\gamma))$ cannot be zero.

In order to show that the respective quantities in (1)–(3) do not vanish, these formal power series in x are skillfully and laboriously decomposed into appropriate parts. The following elementary principles are then applied to these decompositions: (4) If $I \neq 0$ is integral, and Q is a sum of terms of negative degrees, $I+Q$ is not zero. (5) If $I \not\equiv 0 \pmod{F}$ then $I \neq 0$ (this principle occurs in several variants). (6) If D is not zero and is of smaller degree than P, it is not divisible by P.

The general transcendency theorem calls of course for arbitrary polynomials f and possibly fractional arguments γ. Generality is achieved on the basis of: (7) the (known) fact that every polynomial divides a "linear" polynomial; (8) the multiplication theorem for the ψ-function; (9) a special lemma (7.1 of the paper) which takes care of the slight restrictions on f mentioned under (3). *M. A. Zorn.*

Perlis, Sam. A characterization of the radical of an algebra. Bull. Amer. Math. Soc. **48**, 128–132 (1942). [MF 6192]

The author gives the following characterization of the radical H for an associative algebra A/F with a unit element. The radical H is the set of all h in A so that $g+h$ is regular whenever g is regular. This result may be extended to algebras without unit element for multiplication. Following a suggestion of Marshall Hall an element x of a general associative algebra A is called "quasi-regular" if there exists an element y in A with $x+xy+y=0$. It is then proved that r in A lies in the radical if and only if $x+\alpha r$ is quasi-regular for every quasi-regular element x and every α in F.

O. F. G. Schilling (Chicago, Ill.).

Niven, Ivan. Equations in quaternions. Amer. Math. Monthly **48**, 654–661 (1941). [MF 6024]

The following theorem is proved: If D is a non-commutative division algebra with centrum C then every equation of the form $x^m+a_1x^{m-1}+a_2x^{m-2}+\cdots+a_m=0$ $(a_m \neq 0)$ with coefficients in D has a solution in D if and only if C is a real-closed field and D is the algebra of real quaternions over C. The sufficiency of this condition follows from a general result of Ore [Ann. of Math. (2) **34**, 480–508 (1933)] and the necessity was proved by R. Baer. The number of roots of such an equation is also completely discussed, by reducing the problem to the study of a quadratic equation. The result is that there may be infinitely many roots or a finite number, but in the latter case there are at most $(2m-1)^2$. *H. W. Brinkmann* (Swarthmore, Pa.).

Loonstra, F. Bericht über die Konstruktion und die Fortsetzung von Bewertungen. II. Nederl. Akad. Wetensch., Proc. **44**, 806–813 (1941). [MF 5917]

[The first part appeared in the same vol., pp. 286–297; cf. these Rev. **3**, 104.] An abstract of some of the literature on the problem of extension of a non-Archimedean valuation of a field k to a valuation of an algebraic extension K of k. *N. Jacobson* (Chapel Hill, N. C.).

Kaplansky, Irving. Maximal fields with valuations. Duke Math. J. **9**, 303–321 (1942). [MF 6870]

The structure theorems for fields complete in a discrete valuation are generalized here to fields K maximal in an arbitrary valuation V. If V maps K on the ordered Abelian value group Γ, with $\mathfrak{K}$ the associated field of residue classes (mod V), K is called maximal if any extension of its valuation to a larger field involves a proper extension of either the value group Γ or the residue class field $\mathfrak{K}$. It is shown that K is maximal if and only if it contains a limit for each well-ordered sequence of elements pseudo-convergent in the sense of Ostrowski. In the case when $\mathfrak{K}$ has characteristic p, examples show that a restrictive hypothesis is necessary for the structure theorems. A suitable hypothesis is $p\Gamma=\Gamma$, and each polynomial $\sum a_i x^{p^i}$ with $a_i \in \mathfrak{K}$ has a root in $\mathfrak{K}$. The structure theorems state that each field K has a unique least maximal extension, and that a maximal field K with the same characteristic as $\mathfrak{K}$ is a power series field (with a factor set). Finally, a maximal field K is uniquely determined by $\mathfrak{K}$, Γ and V_p, under the additional assumption that every element of $\mathfrak{K}$ has an nth root in $\mathfrak{K}$, for every n.

S. MacLane (Cambridge, Mass.).

Levi, Howard. Composite polynomials with coefficients in an arbitrary field of characteristic zero. Amer. J. Math. **64**, 389–400 (1942). [MF 6440]

A polynomial $F(x)$ is composite if it is of the form $F(x)=P(Q(x))$, where $P(x)$, $Q(x)$ are polynomials of degree greater than 1; this is written $F=P\cdot Q$. In case we have $F=P_1\cdot P_2\cdot\cdots\cdot P_r=Q_1\cdot Q_2\cdot\cdots\cdot Q_s$, where P_i, Q_i are not composite, Ritt has shown [Trans. Amer. Math. Soc. **23**, 51–66 (1922)] that $r=s$ and that the degrees of the two sets of polynomials are the same (except possibly for the order in which they occur). Furthermore, Ritt showed how the general case can essentially be reduced to the case $r=s=2$ and found that $r=s=2$ only in two easily spotted cases. Ritt's proofs made use of the monodromic groups of the polynomials involved and held if the field of the coefficients is that of all complex numbers. Engstrom proved Ritt's first result by purely algebraic means and incidentally showed that the coefficient field can be any field with characteristic zero [Amer. J. Math. **63**, 249–255 (1941); these Rev. **2**, 242]. In the present paper the author uses purely algebraic means to prove Ritt's second result, and this is thus also seen to be true if the field of coefficients is any field of characteristic zero. *H. W. Brinkmann.*

Levi, Howard. On the structure of differential polynomials and on their theory of ideals. Trans. Amer. Math. Soc. **51**, 532–568 (1942). [MF 6552]

Let p be a positive integer and A a differential polynomial in the unknown y. The author obtains a simple criterion for the membership of A in the differential ideal generated by y^p. In particular, if A is a power product in y and its derivatives, A will belong to the ideal if the degree of A exceeds a quantity which depends in a simple manner on the weight of A. These results are applied in the second part of the paper, which studies the manner in which the algebraic structure of a differential polynomial reflects the nature of the decomposition of the manifold of the polynomial. The paper terminates with a study of the ideal generated by uv. This ideal is shown to be neither the intersection nor the product of two ideals whose manifolds are those of u and v, respectively. *J. F. Ritt.*

Kolchin, E. R. On the basis theorem for differential systems. Trans. Amer. Math. Soc. **52**, 115–127 (1942). [MF 6997]

The chief result is a basis theorem for systems of differential polynomials with coefficients in a given differential

ring $\mathfrak{R}$. Hitherto the coefficients have been taken in a differential field of characteristic zero. The ring $\mathfrak{R}$, which may have a positive characteristic, is subjected to a condition of regularity which is satisfied, to take a very special case, if $\mathfrak{R}$ is the totality of rational integers; $\mathfrak{R}$ is said to have a basis theorem if every infinite system σ of elements of $\mathfrak{R}$ has a finite subset ϕ which is such that the perfect ideal generated by ϕ contains σ. It may be noted that the perfect ideal may contain elements which are not roots of elements in the differential ideal generated by ϕ. Let $\mathfrak{R}\{y_1, \cdots, y_n\}$ be the totality of differential polynomials (ordinary or partial) in $y_1, \cdots, y_n$ with coefficients in $\mathfrak{R}$. It is proved that, if $\mathfrak{R}$ contains a unit element, is regular and has a basis theorem, $\mathfrak{R}\{y_1, \cdots, y_n\}$ has a basis theorem. Some information is obtained in regard to the number of "shufflings" required for the generation of an infinite system from a basis.

J. F. Ritt (New York, N. Y.).

Ritt, J. F. Complete difference ideals. Amer. J. Math. 63, 681–690 (1941). [MF 5615]

The author continues previous investigations on difference polynomials and difference ideals [Ritt and Raudenbush, Trans. Amer. Math. Soc. 46, 445–452 (1939); these Rev. 1, 101]. The paper contains various examples pointing out the difference between differential ideals and difference ideals. To every difference ideal $\mathfrak{A}$ over a difference field F there exists a manifold $\mathfrak{M}$ consisting of the totality of solutions of the difference polynomials in $\mathfrak{A}$ in some suitable extension of F. For differential ideals it has been shown that, if the manifold $\mathfrak{M}$ of an ideal $\mathfrak{S}$ is composed of two manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with no solution in common, then $\mathfrak{S}$ has a unique representation as the product of two ideals whose manifolds are $\mathfrak{M}_1$ and $\mathfrak{M}_2$. The author shows that this result does not hold in general for difference polynomials, but the theorem holds if one considers the complete difference ideals, characterized by the property that some power of the transforms of each element in the ideal again belongs to it.

O. Ore (New Haven, Conn.).

Rachevsky, P. Les problèmes les plus simples de "l'algèbre quasi-commutative" en connexion avec la théorie des valeurs caractéristiques des opérateurs différentiels. Rec. Math. [Mat. Sbornik] N.S. 9 (51), 511–544 (1941). (French. Russian summary) [MF 5499]

The author studies noncommutative polynomial rings in which the coefficients are complex numbers commuting with the variables $x_1, x_2, \cdots, x_n$. The quasi-commutative polynomial rings are characterized by the property that for each degree there shall exist exactly as many linearly independent products of variables as in the commutative case. In this paper the case of quasi-commutative polynomials in two variables is studied in detail. One finds as a necessary condition for two variables x and y to define a quasi-commutative ring that they satisfy a quadratic relation

$$a_{11}x^2+a_{12}xy+a_{21}yx+a_{22}y^2+2a_1x+2a_2y+a=0.$$

In a basis (x, y) for the ring the element y is said to be critical if in the fundamental relation $a_{11}=0$. This is a property invariant under linear transformations on x. By means of this concept all types of quasi-commutative rings in two variables are classified into eight types and it is shown that they all define quasi-commutative polynomials. This first part of the paper is closely related to results previously obtained by Littlewood [Proc. London Math. Soc. (2) 35, 200–240 (1933)].

The second part of the paper is concerned with certain factorization properties of quasi-commutative polynomials. The polynomial $P(x, y)$ is said to be reducible (mod $L(x, y)$) when there exists a polynomial $Q(x, y)$ which does not have $L(x, y)$ as a right factor, such that $P(x, y)Q(x, y) = R(x, y)L(x, y)$. For polynomials $P(x, y)$ of first and second degree a method is given to find all factors $L(x, y)$. These investigations have applications to an algebraic theory of differential operators as will be shown in a continuation of the paper.

O. Ore (New Haven, Conn.).

Vandiver, H. S. On formal exponential differentiation in rings. Proc. Nat. Acad. Sci. U. S. A. 28, 24–27 (1942). [MF 6008]

The author considers a ring $\mathfrak{A}$ of polynomials $f(x)=\sum a_i x^{\alpha_i}$, where the a_i and the α_i belong to a commutative ring R containing the natural integers. Addition and multiplication in $\mathfrak{A}$ are defined in the obvious way. The derivation D in $\mathfrak{A}$ is defined by $D(\sum a_i x^{\alpha_i})=\sum a_i \alpha_i x^{\alpha_i}$. Then by using Leibniz's formula for the derivative of a product, the following theorem may be proved [cf. Vandiver, Bull. Amer. Math. Soc. 43, 420–421 (1937)]: Let R be the ring of algebraic integers of an algebraic number field and set

$$f_{n_i}(\alpha_1, \cdots, \alpha_{k_i})=\sum_{r=1}^{k_i} \alpha_r a_r^{n_i}, \qquad i=1, \cdots, s,$$

where the a's and the α's are in R and the n's are rational integers not less than 0. Further let there be a rational integer $d>0$ such that $a_r^d \equiv 1 \pmod{\mathfrak{m}}$ for a fixed ideal $\mathfrak{m}$. Also let $\beta_1+\cdots+\beta_s \equiv 0 \pmod{\mathfrak{m}}$, β_i in R, and suppose that $\mathfrak{a}_i$ is the ideal $(a_{1i}, \cdots, a_{k_i i})$, each of the a's having a factor not equal to (1) in common with $\mathfrak{m}$. Then

$$(f_{n_1}+\cdots+f_{n_s}) \equiv 0 \pmod{(\mathfrak{m}^l, \mathfrak{a}_1, \cdots, \mathfrak{a}_s)},$$

where it is understood here that the left hand member is to be expanded by the multinomial theorem and then $f_{n_i}^t$ is to be replaced by $\beta_i^t f_{n_i+td}^{(t)}(\alpha_1, \cdots, \alpha_{k_i})$.

N. Jacobson (Chapel Hill, N. C.).

Albert, A. A. Division algebras over a function field. Duke Math. J. 8, 750–762 (1941). [MF 5958]

Let A be a simple algebra with center F and let L be a subfield of F. Then a field W over L is called a splitting field over F of A if a composite of W and F is a splitting field in the ordinary sense. The author considers the question as to what can be said about the least degree of a splitting field W that is "constant," that is, algebraic over L. If L has a finite number of elements and $F=L(x)$ then the following theorem is proved: Let m and n be any two integers such that m is a factor of n and every factor of n divides m. Then there exists a division algebra D of degree m over its center $F=L(x)$ such that the least degree of constant splitting fields W over L is n. These conditions are also necessary for the least degree of a splitting field W over L for any division algebra with center F of the type indicated. For any field L that has at least two unequivalent quadratic extensions, there exists a D of degree two over its center $L(x)$ such that no quadratic constant field W over L splits D. If we add the hypothesis that the characteristic of L is two then there exists a D of degree 2^e over $L(x)$ with no constant splitting field of degree 2^e.

N. Jacobson (Chapel Hill, N. C.).

MacLane, Saunders and Schilling, O. F. G. A general Kummer theory for function fields. Duke Math. J. 9, 125–167 (1942). [MF 6347]

The authors generalize the classical notion of extension by radicals of a field F in the following way: F being considered as a function field over a field of constants $\mathfrak{F}$ which is relatively algebraically closed in F, a quantity r is called a radical over F if $r^m=ax$ for some exponent m, with a constant, $x\in F$. An extension of F is called radical if it is generated by radicals; any given algebraic extension of F contains a maximal radical extension. Only separable (and mainly normal) extensions are considered. A divisor in F being defined as a coset of F^* modulo $\mathfrak{F}^*$ (where the * indicates the multiplicative group of the elements not equal to 0), a fractional divisor is a formal expression of the form $d^{1/m}$, d a divisor. To every radical over F there is associated a fractional divisor. The fractional divisors which are associated to the radicals contained in a field $K\supset F$ form a group (R), and K is a radical extension if and only if $[(R):(F)]=[K:F(\mathfrak{R})]$, where $\mathfrak{R}$ is the field of constants of K. Moreover, if K is normal, $(R)/(F)$ is isomorphic to the group of crossed characters in $\mathfrak{R}$ of the Galois group G of K/F (a crossed character satisfying the functional equation $C_\sigma(C_\tau)^\sigma=C_{\sigma\tau}$).

The authors study the structure of the Galois group of a normal radical extension K/F. The main tool is the following theorem: the Galois group of $K/F(\mathfrak{R})$ is isomorphic to the group of characters of $(R)/(F)$. A set of necessary and sufficient conditions is given in order that a finite group G be the Galois group of a radical extension K/F with a given field of constants $\mathfrak{R}$: (a) G must be homeomorphic to the Galois-group Γ of $\mathfrak{R}/\mathfrak{F}$; (b) if $\Gamma=G/H$, H must be Abelian of exponent e prime to the characteristic of F; (c) $\mathfrak{R}$ must contain the eth roots of unity; (d) the group of characters of H must be isomorphic to a group of fractional divisors of F; (e) every character C of H in $\mathfrak{R}$ must satisfy the conditions $C(S^\sigma)=(C(S))^\sigma$, $C(S_{\sigma,\tau})\omega 1$, where σ,τ run over Γ, S over H and $S_{\sigma,\tau}$ is a factor-system of Γ in H corresponding to the extension G of H by Γ.

A number of applications to special cases are given. An interesting negative result is that the principal ideal theorem cannot be generalized for function fields of one variable over a finite field of constants. *C. Chevalley.*

Schneider, Theodor. Zur Theorie der Abelschen Funktionen und Integrale. J. Reine Angew. Math. 183, 110–128 (1941). [MF 6842]

Suppose that $F(x,y)=0$ is an absolutely irreducible polynomial of genus q with algebraic numbers as coefficients. Let

$$\mathfrak{u}=\int_e^{(x,y)}\mathfrak{R}(x,y)dx,\quad \mathfrak{R}(x,y)=\{R_1(x,y),\cdots,R_q(x,y)\},$$

be a basis of the integrals of first kind on $F(x,y)=0$, so that the $R_i(x,y)$ have algebraic coefficients. Letting

$$\mathfrak{U}=\sum_{l=1}^{q}\int_{e_l}^{(x_l,y_l)}\mathfrak{R}(x,y)dx,$$

the author considers the Abelian functions $A(\mathfrak{U})$ which are rational symmetric functions of the q couples (x_i,y_i) with algebraic numbers as coefficients. The two main results of the paper are the following: "Theorem I. If $p+1$ functions $A(\mathfrak{U})$ possess q common periods which are linearly independent in the field of all algebraic numbers, then the functions are algebraically dependent. Theorem II. If the integrand of an integral of the first or second kind is an algebraic function with algebraic numbers as coefficients, then not all periods of the integral are algebraic." The first result is obviously a nontrivial refinement of a well-known fact on Abelian functions. For the proof the author has to study the coefficients of the local inversion problem. First it is shown that the R_i may be selected so as to have algebraic coefficients for their developments relative to x_i-c_i. Moreover, these coefficients can be selected to have bounded denominators and "uniform" upper bounds for all their conjugates. Then comparison of coefficients gives estimates for the local inversion. At various places it is necessary to select the lower limits c_i appropriately so as to stay away from poles, etc. After these preparations the author succeeds in proving both theorems by the method of Cauchy's integral theorem in q space. The latter was used previously by the author and Gelfand in connection with solving Hilbert's problem in the transcendency of α^β, α and β general algebraic numbers. *O. F. G. Schilling* (Chicago, Ill.).

Deuring, Max. Zur Theorie der Moduln algebraischer Funktionenkörper. Math. Z. 47, 34–46 (1940). [MF 6295]

Let k be an arbitrary algebraically closed field of characteristic p. Suppose that K/k is a function field of genus g. Then K is the field of $\sum a_{rs}x^ry^s$ of sufficiently large order n. The arguments of the author require a valid generalization of the theory of modules for curves of genus g to arbitrary coefficient fields. In other words, (1) the fields of genus g are in 1-1 correspondence (except for a subvariety) with the points of a d-dimensional variety $M_g=\{m\}$ over k, $d=3g-3+\rho$, ρ the number of parameters for the automorphism group of K/k; (2) the curves f for variable K determine an irreducible variety $I_{n,g}=\{a_{rs}\}$ over k so that M_g is a rational transform of $I_{n,g}$; (3) the varieties M_g, $I_{n,g}$ and the correspondence $I_{n,g}\to M_g$ can be expressed by equations with coefficients in the prime field P of k. Using van der Waerden's theory of specialization, it is then shown that the field of constants k' of a general field K_n is an algebraic extension of $k(\{m\})$, $\{m\}$ a general point of M_g. Moreover, the coefficients of the general f determine an algebraic extension over $P(\{m\})$ which may be taken to be separable. The author proves the existence of a defining equation f for the general field K_n so that the coefficients of f are rational functions of the modules m with coefficients in P. On specializing $\{m\}$ to $\{m_0\}$ a defining equation f_0 of a special field K_0 is obtained. Both f and f_0 are separable with respect to at least one of the variables. These facts depend for $g>2$ on the assertion that a general field K has no other automorphism but the identity. This crucial theorem is not proved in the paper. The author suggests a proof in the introduction. For the completion of the results for $g=1,2$ the classical results on projective invariants for binary forms of degree 4, 6, respectively, are drawn into the discussion. *O. F. G. Schilling* (Chicago, Ill.).

Deuring, Max. Invarianten und Normalformen elliptischer Funktionenkörper. Math. Z. 47, 47–56 (1940). [MF 6296]

This paper treats of the invariants and normal forms for elliptic fields K over an arbitrary field k of characteristic p. The author defines two fields K_1/k_1, K_2/k_2 to belong to the same type if $K_1\cup k^*=K_2\cup k^*$, k^* the algebraic closure of k. It is assumed that the genus of a field remains unchanged upon extension of the coefficient field. This statement is true if k is a perfect field; however, for general k, the genus may

be altered upon extension of k. The author omits a careful definition of the genus of elliptic fields. Assuming the perfectness of k it is then true that each K/k can be described by a normal form $y^2-y+\alpha xy=x^3$, $\alpha^3\neq -27$ relative to an algebraic extension k' of k. The invariant j of the field is equal to $-\alpha^3(\alpha^3+24)^3/(\alpha^3+27)$ and always $k\supseteq P(j)$, where P is the prime field of k. Conversely, for given j there exists a field K with j as absolute invariant. The discussion evolves around the invariance of the cross ratio

$$[(\xi_1-\xi_3)/(\xi_2-\xi_3)][(\xi_1-\xi_4)/(\xi_2-\xi_4)]^{-1}$$

for the four branch points $x=\xi_1, \cdots, \xi_4$ on a curve $y^2=\rho\Pi_{\nu=1}^{4}(x-\xi_\nu)$ for some $K\cup k'$ and given K/k. Only for $p=2$ is it necessary to depart from the classical discussion of the projective invariants of binary cubics and quartics.

O. F. G. Schilling (Chicago, Ill.).

Schmid, Hermann Ludwig. Zur Meromorphismentheorie der elliptischen Funktionenkörper. Math. Z. **47**, 399–421 (1941). [MF 6726]

Let $K=k(x, y)$ be a field of elliptic functions with the defining equation $F(x, y)=0$ in normalized form over the arbitrary algebraically closed field k. Suppose that $\mathfrak{o}$ is the pole of x, y and μ, ν, ρ meromorphisms of K. Then

$$x\mu-x\nu \cong [\mathfrak{o}(\mu+\nu)\mathfrak{o}(\mu-\nu)][\mathfrak{o}\mu\mathfrak{o}\nu]^{-2},$$

as was first shown by Hasse [J. Reine Angew. Math. **175**, 55–62, 69–88, 193–208 (1936)]. The author observes the analogy of this divisor equivalence with the formula

$$\wp(u_1)-\wp(u_2)=-[\sigma(u_1+u_2)\sigma(u_1-u_2)][\sigma(u_1)\sigma(u_2)]^{-2}$$

of the classical theory of elliptic functions. This remark leads to the formulation and proof of the abstract analogues to some classical identities involving the $\wp$ and σ functions. Of the numerous new results we quote the translation of

$$\begin{vmatrix} 1 & \wp(u_1) & \wp'(u_1) \\ 1 & \wp(u_2) & \wp'(u_2) \\ 1 & \wp(u_3) & \wp'(u_3) \end{vmatrix} = 2\frac{\sigma(u_1+u_2+u_3)\sigma(u_1-u_2)\sigma(u_1-u_3)\sigma(u_2-u_3)}{[\sigma(u_1)\sigma(u_2)\sigma(u_3)]^3}$$

into

$$\begin{vmatrix} 1 & x\mu & y\mu \\ 1 & x\nu & y\nu \\ 1 & x\rho & y\rho \end{vmatrix} \cong \frac{\mathfrak{o}(\mu+\nu+\rho)\mathfrak{o}(\mu-\nu)\mathfrak{o}(\mu-\rho)\mathfrak{o}(\nu-\rho)}{[\mathfrak{o}\mu\mathfrak{o}\nu\mathfrak{o}\rho]^3}.$$

The methods of proof are those used by Hasse [loc. cit.]. The proof of the fundamental identity

$$[x(\mu+\nu)-x(\mu-\nu)][x\mu-x\nu]^2=F_y(x\mu, y\mu)F_y(x\nu, y\nu)$$

goes back, in principle, to a computation in the book of Hensel and Landsberg [Theorie der algebraischen Funktionen, Teubner, Leipzig, 1902, pp. 681–685].

O. F. G. Schilling (Chicago, Ill.).

NUMBER THEORY

***Candy, Albert L. Supplement to Pandiagonal Magic Squares of Prime Order.** Published by the author, Lincoln, Neb., 1942. iii+30 pp.

This supplement generalizes the method of the author's "Pandiagonal Magic Squares of Prime Order" [Lincoln, Neb., 1940; these Rev. **1**, 290]. The number of known squares of order 11 and class II is thus increased from $3\cdot 10^{14}$ to $3\cdot 10^{15}$. (This is still less than the number of such squares of class I, which exceeds 10^{16}.) The Supplement closes with an addendum to the author's "Pandiagonal Magic Squares of Composite Order" [Lincoln, Neb., 1941; these Rev. **2**, 247]. *H. S. M. Coxeter* (Toronto, Ont.).

Vijayaraghavan, T. On Jaina magic squares. Math. Student **9**, 97–102 (1941). [MF 6473]

The author states some known properties of pandiagonal magic squares of order 4, and indicates a simple way of writing any such square. He also discusses squares having similar properties but in which the numbers are not consecutive integers. *R. J. Walker* (Ithaca, N. Y.).

Stern, Erich. Über eine zahlentheoretische Methode zur Bildung und Anzahlbestimmung neuartiger lateinischer Quadrate. Bull. Sci. École Polytéch. Timişoara **10**, 101–131 (1941). [MF 6083]

This paper is concerned with pandiagonal Latin squares of odd prime order, $n=2m+1$. A "cyclic" square of such an order is obtained by taking the entry in the $(\alpha+1)$th column and $(\beta+1)$th row to be $\alpha+q\beta$, reduced mod n, where $1<q\leqq m$. Then each of the residues $0, 1, \cdots, n-1$ occurs once in each column, once in each row, and once in each diagonal. By interchanging any two rows, we obtain another Latin square. But this is no longer pandiagonal; for example, the main diagonal has acquired, in the rows considered, two numbers which already occur elsewhere, namely, in two other rows. We may attempt to restore the distribution of numbers in the diagonal by interchanging these other rows too. But the same trouble will then be caused by two further numbers, compelling us to interchange yet another pair of rows. We continue thus until the entries in the main diagonal are all different, as at the beginning. Suppose this requires altogether s interchanges. If $s=m$, we merely have another cyclic square. But if $s<m$ the final square is only "half-cyclic," although it is still pandiagonal. Latin squares of this new kind can be formed whenever m is composite, that is, for $n=13, 17, 19, 29, 31, 37, 41, 43, \cdots$. The process is generalized in various ways.

H. S. M. Coxeter (Toronto, Ont.).

Ramanathan, K. G. On Demlo numbers. Math. Student **9**, 112–114 (1941). [MF 6475]

This note on digital properties of numbers is a contribution to the problem of describing the digits of the product of two factors in terms of the digits of the factors. In the present case one factor N is an arbitrary number of k digits while the other factor is $(10^n-1)/9$ and thus consists of the digit 1 repeated n times. If m denotes the greatest integer less than $N/9$ and $a=N-9m$, then the product in question begins with the digits of m and ends with the digits of $a(10^k-1)/9-m$, while the $n-k$ remaining digits consist entirely of a's. The special case of the square of $(10^n-1)/9$ is also considered and illustrated. *D. H. Lehmer.*

Chatterjee, B. C. On irrationality. Bull. Calcutta Math. Soc. **32**, 69–71 (1940). [MF 6158]

The author proves the following interesting result. Let a_i denote the ith digit after the decimal point in the decimal expression for a number A. If for some constant n, and any prime number p of s digits, the digits of p will occur in proper order (but not necessarily contiguous) among the first $[4.19s+n]$ digits a_i, then A is an irrational number.

P. Franklin (Cambridge, Mass.).

Aude, H. T. R. Primitive integral triangles. Nat. Math. Mag. 16, 280–283 (1942). [MF 6354]

This paper is concerned with the parametric representation of the sides a, b, c of an integral triangle in which the cosine of the angle C between a and b has a specified rational value r/s. It is found that

$$a : b : c = 2n(ms-nr) : s(m^2-n^2) : (m^2s-2mnr+n^2s),$$

where m, n are a pair of coprime integers. A similar triangle is given by $m'=(m+n)s-2nr$, $n'=(m-n)s$. The cases of $C=120°$, $60°$ and $\arccos(-1/4)$ are considered in detail.

D. H. Lehmer (Berkeley, Calif.).

Griffiths, Lois W. Universal functions of polygonal numbers. Amer. Math. Monthly 49, 107–110 (1942). [MF 6244]

This is an expository paper on the subject of representing numbers as sums of polygonal numbers beginning with Fermat's problem and its development by Lagrange, Cauchy and recently by Dickson and the author. The reviewer takes exception to the statement that Gauss first proved that $8n+7$ is not the sum of 3 squares. This obvious fact was known to Diophantus. *D. H. Lehmer* (Berkeley, Calif.).

Griffiths, L. W. A note on representation by polygonal numbers. Bull. Amer. Math. Soc. 48, 122–124 (1942). [MF 6190]

The problem considered in this paper is the representation of all positive integers by the linear combination

$$a_1f(x_1)+a_2f(x_2)+\cdots+a_nf(x_n), \tag{1}$$

where the a's are positive integers, the x's are nonnegative integers, and $f(x)=f_m(x)=x+m(x^2-x)/2$. For $m\geqq 3$, the least value N of n for which such a linear combination (1) exists is proved to be the greatest integer less than $\log_2(8m+8)$. Such a minimal combination exists uniquely (except for rearrangement of the terms) if and only if $m=3, 4, 2^{N-2}-2, 2^{N-2}-1$. *D. H. Lehmer.*

Kanold, Hans-Joachim. Untersuchungen über ungerade vollkommene Zahlen. J. Reine Angew. Math. 183, 98–109 (1941). [MF 6841]

Let $F_m(x)$ be the mth cyclotomic polynomial, where $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with $p_1<p_2<\cdots<p_k$. The author improves a well-known theorem of Kronecker as follows. If the integer q is a divisor of m and a divisor of $F_m(x)$ for an arbitrary integer x, then

$$q\equiv p_k\equiv 1 \pmod{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}}}.$$

This result, however, is not new [cf. J. Petersen, On the Sum of Quadratic Residues and the Distribution of Prime Numbers of the Form $4n+3$, Anniversary Volume, University of Copenhagen, 1907 (Danish); T. Nagel, Norsk Mat. Tidsskr. 1, 95–98 (1919); Norsk Mat. Forenings Skr. Ser. 1., no. 3, 17 pp. (1921)]. Using this result the author obtains necessary conditions for the existence of odd perfect integers. It is well known that such a number n must have the form $n=p^{\alpha}q_1^{2\beta_1}q_2^{2\beta_2}\cdots q_r^{2\beta_r}$ with $p\equiv\alpha\equiv 1 \pmod 4$. The author proves that n must be divisible by d^4 if d is the greatest common divisor of $2\beta_1+1, 2\beta_2+1, \cdots, 2\beta_r+1$. From this result he obtains that n cannot be perfect if d is divisible by 9, 15, 21 or 33. Moreover, suppose that $2\beta_\rho+1=l^{\gamma}$ $(\rho=1, 2, \cdots, r)$, where l is a prime. If n is perfect, it is necessary that $p\equiv 1 \pmod l$ and $\frac{1}{2}(\alpha+1)\equiv 0$ or $1 \pmod l$. For instance, $n=p^5q_1^6q_2^6\cdots q_r^6$ is not perfect. It was shown by Steuerwald [S.-B. Bayer. Akad. Wiss. 1937, 69–72] that $n=p^{\alpha}q_1^3q_2^2\cdots q_r^2$ is not perfect. The author proves the same for $n=p^{\alpha}q_1^4q_2^4\cdots q_r^4$. *A. Brauer.*

Basu, N. M. On a Diophantine equation. Bull. Calcutta Math. Soc. 32, 15–20 (1940). [MF 6153]

In a recent note Tchacaloff and Karanicoloff [C. R. Acad. Sci. Paris 200, 281–283 (1940); these Rev. 1, 200] have obtained the general parametric solution, in nonzero rational numbers, of the equation $Ax^m+By^n=z^p$, where m, n, p are relatively prime in pairs and A and B are any rational numbers. For example, in the case of the equation $x^2-y^3=z^5$, the solution obtained is

$$(x, y, z)=(a^5b^{10}(a-b)^{-3}, a^5b^7(a-b)^{-2}, a^2b^4(a-b)^{-1}),$$

where the parameters a, b run over all nonzero rationals with $a\neq b$. Solving for a and b in terms of x, y, z gives $a=xy^{-2}z$, $b=x^{-1}yz$, so that the solution is general.

In the present note the method is extended to the general equation

$$A_1x_1^m+A_2x_2^n+\cdots+A_kx_k^l=x_{k+1}^r,$$

where the A's are rational and the integers $m, n, \cdots, l$ are prime to r and are such that any pair of them have the same greatest common divisor g. The general solution is given in terms of the solutions $(y_1, \cdots, y_{k+1})$ of

$$A_1y_1^g+A_2y_2^g+\cdots+A_ky_k^g=y_{k+1}.$$

D. H. Lehmer (Berkeley, Calif.).

Olds, C. D. On the number of representations of the square of an integer as the sum of three squares. Amer. J. Math. 63, 763–767 (1941). [MF 5623]

By using a Liouville identity of J. V. Uspensky [Bull. Acad. Sci. USSR 20, 327–348 (1926)] an elementary proof is obtained of the formula for the number of solutions of $n^2=x^2+y^2+z^2$. *G. Pall* (Montreal, P. Q.).

Barnett, I. A. and Mendel, C. W. On equal sums of squares. Amer. Math. Monthly 49, 157–170 (1942). [MF 6392]

The authors first solve the equation $\sum_1^n x_i^2=\sum_1^n y_i^2$ by writing it in the form $\sum_1^n(x_i-y_i)\cdot(x_i+y_i)=0$; this is a homogeneous linear Diophantine equation in x_i+y_i with coefficients x_i-y_i. If x_i-y_i are introduced as parameters one can solve for x_i+y_i in terms of these and the additional parameters that arise in the solution of such a linear equation. Finally one can solve for x_i, y_i in terms of all these parameters. This process must be modified considerably to make sure that x_i, y_i are integral; the authors succeed in doing this and thus obtain a general solution in terms of a set of parameters. To get a solution with a minimum number of parameters they first find a simple expression (which appears to be new) for the solution of $\sum_1^n a_ix_i=0$ in terms of $n-1$ parameters.

The result just explained is then used to solve the system $\sum_1^n x_i^2=\sum_1^n y_i^2$, $\sum_1^n x_i=\sum_1^n y_i$; this is done by determining the parameters in the solution of the first equation in such a way as to satisfy the second. Likewise a complete parametric solution of the more general equation $\sum_1^m x_i^2=\sum_1^n y_i^2$ $(m<n)$ is found by determining the parameters in such a way that $n-m$ of the values of y_i are zero. For $m=1$, that is, $y_1^2=\sum_1^n x_i^2$ the results are particularly simple.

H. W. Brinkmann (Swarthmore, Pa.).

Bell, E. T. Selective equations. Ann. of Math. (2) 42, 1029–1036 (1941). [MF 5529]

The author considers the systematic solution of certain general Diophantine systems which are shown to be com-

pletely solvable in a finite number of prescribed steps in terms of a finite (though sometimes extravagantly large) number of independent integer parameters. It is assumed that out of a finite number of arbitrary integers $n_1, n_2, \cdots, n_s$ it is possible to select the least and the greatest, denoted by $(n_1, n_2, \cdots, n_s)'$ and $[n_1, n_2, \cdots, n_s]'$, respectively. These operators $(\ )'$ and $[\ ]'$ are called the two simple selective operators. Compound selective operators are exemplified by such expressions as

$$(n_1, (n_2, n_3)', n_4)', \quad (n_1, [n_2, n_3], n_4, (n_5, n_6, n_7)')'.$$

A selective system is a finite system of the type

$$\sum A_i x_i + \sum a_i X_i' = \cdots = \sum B_i y_i + \sum b_i Y_i',$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\sum C_i x_i + \sum c_i Z_i' = \cdots = \sum D_i w_i + \sum d_i W_i',$$

where the X', Y', Z', W' are abbreviations for the results of certain specified compound selective operators on unknowns $x_{ik}, y_{ik}, \cdots, w_{ik}$ and where x_i, y_i, w_i, z_i are also unknown. The other letters are given. Such a system obviously has a dual in which $(\ )'$ and $[\ ]'$ are replaced by the greatest common divisor and least common multiple operators $(\ )$ and $[\ ]$, and addition in the above system is replaced by multiplication. This dual system is solved explicitly in terms of parameters subject to certain G.C.D. conditions. Compound operators are first reduced to simple ones and the resulting simple system is solved recursively. From this the solution of the original system can be written down, the complete solution being separated into sets of solutions by the dual of the G.C.D. conditions. *D. H. Lehmer.*

Simmons, H. A. Classes of maximum numbers associated with two symmetric equations. Bull. Amer. Math. Soc. **48**, 295–303 (1942). [MF 6409]

One of the Diophantine equations studied in this paper is

$$\sum_{n,\,n-1} (1/x) + \sum_{i=1}^{m} a_i [\pi(x)]^{-i} = k,$$

where $\sum_{i,j}(1/x)$ stands for the elementary symmetric function of the jth order of the i reciprocals $1/x_p$, $\pi(x) = x_1 x_2 \cdots x_n$, $a_i \geqq 0$ and $k = b/[(c+1)b-1]$, b and c positive integers. The second of the equations, not written here, includes as special instances certain equations previously considered by the author [Simmons and Block, Duke Math. J. **2**, 317–339 (1936)]. The solution $x = (x_1, x_2, \cdots, x_n)$ of either of these equations obtained by minimizing the variables $x_1, x_2, \cdots, x_{n-1}$ in this order, one at a time, among the positive integers is called the Kellogg solution w, while an E-solution is one in which $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, where $x_1, x_2, \cdots, x_{n-1}$ are positive integers. He proves that for either of these equations (1) the Kellogg solution contains the largest number that exists in any E-solution, and (2) if $X \neq w$ is an E-solution, then $P(x) < P(w)$, where $P(x) = P(x_1, x_2, \cdots, x_n)$ is a symmetric polynomial, not a constant, having no negative coefficients. Both of these results have already been announced by the author for the more special equations referred to above. *I. A. Barnett.*

Pillai, S. S. On numbers of the form $2^a \cdot 3^b$. I. Proc. Indian Acad. Sci., Sect. A. **15**, 128–132 (1942). [MF 6516]

Ramanujan asserted that, if $N(x)$ is the number of integers of the form $2^a 3^b$ which are less than x, then approximately

$$(1) \qquad N(x) = \frac{\log 2x \log 3x}{2 \log 2 \log 3}.$$

Let $\Delta(x) = N(x) - F(x)$, where $F(x)$ is the right member of (1). The order of $\Delta(x)$ may be obtained from the Hardy-Littlewood results on the problem of lattice points in a right angled triangle. In this paper formulas for $N(2^r \cdot 3^s)$ and $\Delta(2^r \cdot 3^s)$ are derived by the ingenious use of recursion formulas connecting $N(x)$ and $\Delta(x)$ with $N(\frac{1}{2}x)$, $N(\frac{1}{3}x)$ and $\Delta(\frac{1}{2}x)$, $\Delta(\frac{1}{3}x)$, respectively. A table is given of integers of the form $2^a \cdot 3^b$ up to $2^{10} \cdot 3^6$. *R. D. James.*

Pillai, S. S. and George, Alleyamma. On numbers of the form $2^a \cdot 3^b$. II. Proc. Indian Acad. Sci., Sect. A. **15**, 133–134 (1942). [MF 6517]

From the results and table of part I [see the preceding review] with additional tables it is shown that $-1.9 < \Delta(x) < 2.4$ for $x \leqq 2^{1000}$. *R. D. James* (Saskatoon, Sask.).

Atkinson, F. V. A divisor problem. Quart. J. Math., Oxford Ser. **12**, 193–200 (1941). [MF 6544]

If $d_k(n)$ is the number of ways of expressing n as the product of k factors then Piltz's formula is $\sum_{n \leq x} d_k(n) = x f_k(\log x) + \Delta_k(x)$, where $f_k(x)$ is the residue of $\zeta^k(s) x^{s-1}/s$ at $s=1$ and $\Delta_k(x) = O(x^\alpha)$. It is shown that, if α_k denotes the lower bound of the α for which the estimate holds, then $\alpha_3 \leqq 37/75$. This is somewhat better than the results $\alpha_3 \leqq 43/87$ [Walfisz] and $\alpha_k \leqq \max(\frac{1}{2}, (k-1)/(k+2))$ [Hardy and Littlewood]. *H. S. Zuckerman* (Seattle, Wash.).

Sen, D. N. A problem on "factorisatio numerorum." Bull. Calcutta Math. Soc. **33**, 1–8 (1941). [MF 6164]

Let $f(n)$ be the number of representations of n as the product of factors greater than 1, two representations being equal if and only if they involve the same factors in the same order. The author derives several recurrence formulas and generating functions for $f(n)$ when the number of prime factors of n is specified. If all prime factors are simple, one of the recurrence formulas of the reviewer [Acta Arith. **2**, 134–144 (1936), formula (2.4)] fails and the author gives the necessary modifications. If $n = p_1 p_2 \cdots p_\nu$, he shows that $f(n) = \sum_{k=0}^{\nu} k! 2^{\nu-k-1}$. He also computes

$$f(p^\alpha q^\beta) = 2^{\alpha-1} F(-\beta, \alpha+1, 1, -1) = 2^{\alpha+\beta} \frac{(\alpha+\beta)!}{\alpha!\beta!} F(-\alpha, -\beta, -(\alpha+\beta), \tfrac{1}{2}),$$

where F denotes the usual hypergeometric series. *E. Hille* (New Haven, Conn.).

Vandiver, H. S. Certain congruence criteria connected with Fermat's last theorem. Proc. Nat. Acad. Sci. U. S. A. **28**, 144–150 (1942). [MF 6486]

The author derives congruences that must hold if Fermat's equation $x^l + y^l + z^l = 0$ (l an odd prime) has solutions. They are derived by an extension of a method used previously by the author to obtain similar criteria [Proc. Nat. Acad. Sci. U. S. A. **15**, 43–48 (1929), in particular, p. 45]. As an illustration let xyz be prime to l (the "first case"). Certain principal prime ideals $\mathfrak{p}$, each dividing a prime p, are described, and it is found that there are infinitely many of these. It turns out that $N(\mathfrak{p}) = p = 1 + cl$ and by applying the theorem of reciprocity one gets $(x + y\zeta^s/\mathfrak{p}) = 1$, so that $(x + y\zeta^s)^c \equiv 1 \pmod{\mathfrak{p}}$; here $\zeta^l = 1$, $\zeta \neq 1$, $s = 0, 1, \cdots, l-1$. These congruences are then combined to yield various congruences free from ζ and the resulting congruences then hold modulo p. Similar congruences are derived for the "second case." Finally, by a process of elimination, congruences modulo p are found that do not contain x, y.

H. W. Brinkmann (Swarthmore, Pa.).

Hua, Loo-keng and Min, Szu-hoa. On the number of solutions of certain congruences. Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 113–133 (1940). [MF 5322]

Let $f(x)=a_kx^k+\cdots+a_1x+a_0$, the a_i integral, p a prime, $p\nmid(a_1,\cdots,a_k)$. The number $N(s,p,n)$ of integer solutions $x_1,\cdots,x_s$ of $f(x_1)+\cdots+f(x_s)\equiv n \pmod p$ is $p^{s-1}+p^{-1}E(s)$, where

$$|E(s)|\leqq T(s)=\sum_{\sigma=1}^{p-1}\Big|\sum_{x=1}^{p}e^{2\pi i\sigma f(x)/p}\Big|^s.$$

The object of this paper is to estimate $T(s)$ for large p. From a result of Davenport [J. Reine Angew. Math. **169**, 158–176 (1933)] is deduced $T(s)=O(p^{s-(s-2)/m})$ if $s\geqq 2$ and $k\geqq 3$; here $m=8/3$ if $k=3$, and m is the largest integer of the form 2^σ or $3\cdot 2^\sigma$, and not exceeding k, if $k>3$. (Hence if $s>2$, $N(s,p,n)\sim p^{s-1}$; but this is false if $s=2$.) To obtain an improvement in certain cases, the authors show that $T(2t)=O(p^{2t-\delta})$ when $t=k$, and when $t=2^r[k/2^r]$, where r is a positive integer such that $[k/2^r]=3$, 4 or 5. Then by Hölder's inequality, or directly, follows $T(s)=O(p^{s-(s-\delta)/(t-1)})$ if $2\leqq s\leqq 2t$, $T(s)=O(p^{s-2-(s-2\delta)/m})$ if $s>2t$. The paper contains hundreds of misprints. *G. Pall* (Montreal, Que.).

Hua, Loo-keng. On Waring's problem with cubic polynomial summands. Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 55–83 (1940). [MF 5321]

Hardy and Littlewood [Math. Z. **23**, 1–37 (1925)] proved that almost all positive integers are sums of five positive integer cubes. Davenport [Acta Math. **71**, 123–143 (1939); these Rev. **1**, 5] reduced the number to 4, which is best possible, using a lemma on the number of solutions of $x_1^3+y_1^3+z_1^3=x_2^3+y_2^3+z_2^3$. Hua now proves an extension to all integral-valued polynomials, there being two specific exceptions [cf. previous announcement, C. R. Acad. Sci. Paris **210**, 650–652 (1940); these Rev. **2**, 35]. The second exception $f(x)=ax^3+3bx^2+3cx+d \pmod 9$ with $b^3\equiv a(c+a) \pmod 3$ requires five summands; it was earlier stated incorrectly as $b^3\equiv a(b+a)$. The analytical part is based on Davenport's work. The arithmetical part requires the solvability of $f(x_1)+\cdots+f(x_4)\equiv n \pmod{p^\gamma}$ in integers x_ν such that not every $f'(x_\nu)/p^k$ is divisible by p; here k is the least integer not less than -1 such that $f'(x)/p^k$ is prime to p for some x, and $\gamma=2+(-1)^p+k$. This fails if $p=3$ and $f(x)=x^3-3x$, giving the above exception. An exception requiring seven summands arises from $p=2$. *G. Pall.*

Hua, Loo-keng. On the number of partitions of a number into unequal parts. Trans. Amer. Math. Soc. **51**, 194–201 (1942). [MF 6094]

It is proved that the number $q(n)$ of partitions of n into unequal parts is equal to

$$2^{-\frac{1}{2}}\sum_{k}^{1,3,5,\cdots}A_k(n)\,d\,J_0((i\pi/k)\{\tfrac{1}{2}(n+1/24)\}^{\frac{1}{2}})/dn,$$

where $A_k(n)=\sum_h\omega_{h,k}e^{-2\pi ihn/k}$ (h ranging over $0<h\leqq k$, $(h,k)=1$), and $\omega_{h,k}$ is a certain $24k$th root of unity. The method of Hardy and Ramanujan for $p(n)$ [Proc. London Math. Soc. (2) **17**, 75–115 (1918)] is employed, with modifications due to Rademacher [ibid. (2) **43**, 241–254 (1937)]. *G. Pall* (Montreal, Que.).

Brauer, Alfred. On a problem of partitions. Amer. J. Math. **64**, 299–312 (1942). [MF 6432]

The author considers the Diophantine linear equation

$$a_1x_1+a_2x_2+\cdots+a_kx_k=n,$$

in which the letters are positive integers and the a's are without common factor. The problem discussed is that of determining a bound $F=F(a_1,\cdots,a_k)$ such that for all $n>F$ we are assured of the existence of the k unknown x's. For $k=2$ a bound $F(a_1,a_2)=a_1a_2$ was given by Sylvester. This bound is the best possible. For $k>2$ a bound F is given by $S=a_1a_k+a_2+a_3+\cdots+a_{k-1}$. This is a previously unpublished result of I. Schur. If $d_\nu=(a_1,a_2,a_3,\cdots,a_\nu)$ is the greatest common divisor of the first ν a's, so that $d_1=a_1$, $d_k=1$, then an alternative bound is given by

$$T=\sum_{\nu=2}^{k}a_\nu d_{\nu-1}/d_\nu.$$

It is shown that T is at least as good as S and that the two bounds are equally good when and only when S is the best possible. The question of when T is the best bound is further discussed. In the special case in which the a's are the k consecutive integers from m to $m+k-1$, the best bound is given by the greatest integer not exceeding $\frac{1}{2}(k^2-k-2)+(m+k^2-3)/(k-1)$. *D. H. Lehmer.*

Erdös, P. and Turán, P. On a problem of Sidon in additive number theory, and on some related problems. J. London Math. Soc. **16**, 212–215 (1941). [MF 6615]

S. Sidon [Math. Ann. **106**, 536–539 (1932)] used the letter B_2 to characterize a sequence $a_1<a_2<\cdots$ of positive integers, where the sums a_i+a_j (with $i\leqq j$) are all different. He derived some results about $\Phi(n)$, the maximum number of terms not exceeding n which a B_2 sequence can have. In this paper, Sidon's results are bettered by the following result

$$\frac{1}{\sqrt{2}}\leqq\underline{\lim}\,\frac{\Phi(n)}{\sqrt{n}}\leqq\overline{\lim}\,\frac{\Phi(n)}{\sqrt{n}}\leqq 1.$$

The proof is elementary except for the result that the quotient of consecutive primes tends to 1. The authors remark that it is very likely that $\lim\Phi(n)/\sqrt{n}$ exists but they have not succeeded in proving it. *B. W. Jones.*

Hartman, Philip and Wintner, Aurel. Additive functions and almost periodicity. Duke Math. J. **9**, 112–119 (1942). [MF 6345]

An additive function $f(n)$ (in the number-theoretic sense of this paper) is one which has the property that $f(mn)=f(m)+f(n)$ whenever m and n are relatively prime. Such functions have been shown to have certain almost periodic properties when suitably restricted, and the authors now show that an additive function is almost periodic in the sense of Weyl (W^2 a.p.) if and only if it is bounded. It is also shown that an additive function has a uniform mean value if and only if it is bounded, and a number of illustrations are given to show the difference between the ordinary mean value and the more restrictive uniform mean value which exists only when $[f(m+1)+\cdots+f(m+n)]/n$ approaches a constant limit uniformly for positive m as $n\to\infty$. *R. H. Cameron* (Cambridge, Mass.).

Wintner, Aurel. Prime divisors and almost periodicity. J. Math. Phys. Mass. Inst. Tech. **21**, 52–56 (1942). [MF 6779]

Using a representation by Ramanujan sums the author shows that the function

$$\delta(n)=\sum_{p\mid n}1-\sum_{p^2\mid n}1$$

is almost periodic (B^λ) for every $\lambda>0$ no matter how large.

However, since it is unbounded, it is not almost periodic Bohr, and, in fact, its asymptotic distribution has arbitrarily large jumps. *S. Bochner* (Princeton, N. J.).

Wintner, Aurel. On the prime number theorem. Amer. J. Math. **64**, 320–326 (1942). [MF 6434]

The author points out advantages of the Tauberian method for certain proofs and discussions in the analytic theory of numbers. He then states and proves a generalization of Ikehara's theorem and shows that it unifies and simplifies the proofs of several theorems, including the prime number theorem. *H. S. Zuckerman* (Seattle, Wash.).

Wintner, Aurel. The distribution of primes. Duke Math. J. **9**, 425–430 (1942). [MF 6879]

Let $f(n)$ be an additive arithmetical function such that, for every prime p, (a) $f(p)=1$, (b) $f(p^k)>0$ $(k=2, 3, \cdots)$. Furthermore, let $f_m(x)$ denote the number of integers $j<x$ for which $f(j)=m$. The author proves the asymptotic relationship

$$f_m(x)\sim\frac{x(\log x)^{-1}(\log\log x)^{m-1}}{(m-1)!},\quad m=1, 2, \cdots.$$

The result is first approached heuristically by means of the Poisson probability distribution and then proved rigorously using the prime number theorem and some of its consequences. *M. Kac* (Ithaca, N. Y.).

Ingham, A. E. On two conjectures in the theory of numbers. Amer. J. Math. **64**, 313–319 (1942). [MF 6433]

The conjectures referred to are that $M(x)=\sum_{n\leq x}\mu(n)=O(x^{\frac{1}{2}})$ and $L(x)=\sum_{n\leq x}\lambda(n)\geqq 0$, $x\geqq 2$. A consequence of either of these conjectures is the Riemann hypothesis and the fact that all the zeros of the zeta-function are simple. In this paper it is further shown that the imaginary parts of all the zeros above the real axis must be linearly dependent. It is proved that if the imaginary parts of the zeros are connected by no, or only a finite number of, linear relations with integral coefficients, then $\underline{\lim}\, x^{-\frac{1}{2}}M(x)=-\infty$, $\overline{\lim}\, x^{-\frac{1}{2}}M(x)=+\infty$, $\underline{\lim}\, x^{-\frac{1}{2}}L(x)=-\infty$, $\overline{\lim}\, x^{-\frac{1}{2}}L(x)=+\infty$, as $x\to\infty$. *H. S. Zuckerman* (Seattle, Wash.).

Rademacher, H. A. Correction. Amer. J. Math. **64**, 456 (1942). [MF 6445]

Concerning the author's paper "The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$," Amer. J. Math. **61**, 237–248 (1939).

Rademacher, Hans. The Ramanujan identities under modular substitutions. Trans. Amer. Math. Soc. **51**, 609–636 (1942). [MF 6555]

The identities considered are

$$[1]\qquad \sum_{l=0}^{\infty}p(5l+4)x^l=5\prod_{m=1}^{\infty}(1-x^{5m})^5/\prod_{m=1}^{\infty}(1-x^m)^6$$

and the similar ones involving the moduli 7, 13, 5^2 and 7^2. The author writes these identities in terms of Dedekind's η-function and then subjects them to modular transformations. Thus [1] can be written in the form

$$[2]\qquad \sum_{\lambda=0}^{4}\eta\left(\frac{\tau+24\lambda}{5}\right)^{-1}=5^2\frac{\eta(5\tau)^6}{\eta(\tau)^6}$$

and, under the transformation $\tau\to-\tau^{-1}$, this goes over into a new identity, one form of which expresses certain quadratic relations between the $p(n)$. The author also obtains new proofs of the original identities. For the modulus 5 he multiplies [2] by $\eta(\tau)$ and considers the two members as modular functions belonging to the subgroup $\Gamma_0(5)$. From a knowledge of the transformation equation of $\eta(\tau)$ and a consideration of the behavior of the functions at the parabolic points, he establishes the identity of the two modular functions, and hence proves [1]. The author also shows that the identities with moduli 5^2 and 7^2 can be made to yield certain modular equations when subjected to the transformation $\tau\to-\tau^{-1}$. *H. S. Zuckerman.*

Rademacher, Hans. Trends in research: the analytic number theory. Bull. Amer. Math. Soc. **48**, 379–401 (1942). [MF 6704]

This is an address delivered before the American Mathematical Society. The author limits his subject, for the most part, to a description of significant methods and results in analytic number theory after the work of Landau, Hardy and Littlewood. The discussion is arranged in such a way that it brings out two points that are frequently overlooked. The first is that analytic number theory is not restricted to asymptotic formulas and estimates but that it has another side which is concerned with the derivation of identities, the use of group theoretical arguments, etc. The second point is that analytic number theory is not merely a device for proving number theoretical results with the aid of analysis, but that it is really a thorough fusion of analysis and arithmetic in which the main interest is often as much on the analytical part as on the arithmetical part.

H. S. Zuckerman (Seattle, Wash.).

Carlitz, L. The reciprocal of certain series. Duke Math. J. **9**, 234–243 (1942). [MF 6351]

The author studies the coefficients of the reciprocal of a series of the type $f(u)=\sum_0^\infty(-1)^i(A_i/F_i)u^{p^i}$, where $F_i=[i]F_{i-1}^{p}$, $[i]=x^{p^i}-x$, $F_0=1$; and A_i are arbitrary polynomials in x with coefficients in $GF(p^n)$, $A_0=1$. He sets $u/f(u)=\sum_0^\infty(\beta_m/g_m)u^m$, $(p^n-1\,|\,m)$, where $g_m=F_0^{a_0}F_1^{a_1}\cdots$ $(m=\sum a_ip^i,\ 0\leqq a_i<p)$. Under the assumption that the function inverse to $f(u)$ has an expansion of a certain form he proves a theorem of the von Staudt type for the β_m. The case $A_i=1$ was considered by the author in a previous paper [Duke Math. J. **3**, 503–517 (1937); **7**, 62–67 (1940); cf. these Rev. **2**, 146]. *H. W. Brinkmann.*

Segal, B. Character sums and their application. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 401–410 (1941). (Russian. English summary) [MF 6829]

Let $\chi_1, \cdots, \chi_r$ be nonprincipal multiplicative characters mod p $(\geqq 5)$; let $f_1(x), \cdots, f_r(x)$ be different normalized polynomials of degrees $k_1, \cdots, k_r$, each irreducible mod p. Set $k_1+\cdots+k_r=k$. H. Davenport proved [Acta Math. **71**, 99–121 (1939); these Rev. **1**, 41] that

$$\left|\sum_{x=0}^{p-1}\chi_1(f_1(x))\cdots\chi_r(f_r(x))\right|\leqq(k-1)p^{1-\theta_k},$$

where $\theta_2=\frac{1}{2}$, $\theta_3=\frac{1}{4}$, $\theta_k=3/(2k+8)$ if $k\geqq 4$; and θ_k can be replaced by θ_{k-1} if $\chi_1^{k_1}\cdots\chi_r^{k_r}=\chi_0$. Let $1\leqq q<p-1$, $\rho_k=3/(8k+16)$. Segal treats the partial sum in which x runs only from 0 to q, and proves it is numerically less than $2(r^2-r+2k)^{\frac{1}{2}}p^{1-\rho_k}\log p$. Applications of both Davenport's and Segal's results follow. (i) Let $\chi_1, \cdots, \chi_n$ be any nonprincipal characters mod p, $\epsilon_1, \cdots, \epsilon_n$ be any set of n roots of unity, ϵ_i being an l_ith root of unity. Let $E(\epsilon_1, \cdots, \epsilon_n)$ denote the number of sequences $x+1, x+2, \cdots, x+n$ out

of $1, 2, \cdots, p-1$ for which $\chi_1(x+1)=\epsilon_1, \cdots, \chi_n(x+n)=\epsilon_n$; and $E_q(\epsilon_1, \cdots, \epsilon_n)$ the number of such sequences out of $1, \cdots, q$. By Davenport, $E=p/(l_1 \cdots l_n)+\Theta n(p^{1-\theta_n}+1)$; by Segal, $E_q=q/(l_1 \cdots l_n)+\Theta \cdot 2(n+1)p^{1-\rho_n} \log p$; Θ denotes some number between -1 and $+1$. (ii) The number of sequences out of $1, \cdots, p-1$ such that all of $x+1, \cdots, x+n$ are primitive roots mod p is

$$p(p-1)^{-n}(\varphi(p-1))^n+\Theta 2^{n\sigma}n(p^{1-\theta_n}+1),$$

σ denoting the number of distinct primes dividing $p-1$; hence, for any n, large primes have n successive primitive roots. The number of such sequences out of $1, \cdots, q$ is

$$q(p-1)^{-n}(\varphi(p-1))^n+\Theta(n+1)2^{n\sigma+1}p^{1-\rho_n} \log p.$$

(iii) The number of sequences out of $1, \cdots, p-1$ for which $1 \leqq \operatorname{ind}(x+1) \leqq Q_1, \cdots, 1 \leqq \operatorname{ind}(x+n) \leqq Q_n$, where $Q_1, \cdots, Q_n$ are given positive integers not greater than $p-1$, is given by

$$Q_1 \cdots Q_n(p-1)^{1-n}+\Theta n 2^n(p^{1-\theta_n}+2) \log^n(p-1);$$

similarly for sequences out of $1, \cdots, q$. (iv) If $f(x)$ is a polynomial with integer coefficients, and if $f(x) \not\equiv (g_1(x))^2 g_2(x)$ mod p for any nonconstant $g_1(x)$, then an estimate is found for the number of values of $f(x)$, for $x=0, \cdots, p-1$, or for $x=0, \cdots, q$, such that $f(x)$ is a primitive root mod p, or such that $\chi(f(x))$ has a given value. *G. Pall.*

Gilbarg, David. The structure of the group of $\mathfrak{P}$-adic 1-units. Duke Math. J. **9**, 262–271 (1942). [MF 6867]

The 1-units of a $\mathfrak{P}$-adic number field K are those units ϵ which are congruent to 1 modulo $\mathfrak{P}$. Let K be normal over k, let G be the Galois group of K/k with elements $\sigma, \tau, \cdots$, and let Γ be the group ring of G over the ring of p-adic integers, where p is rational, $\mathfrak{P}p$. For any 1-unit ϵ and element $\zeta=a\sigma+b\tau+\cdots$ of Γ, the hypercomplex power ϵ^ζ is defined by $\epsilon^\zeta=(\sigma\epsilon)^a(\tau\epsilon)^b \cdots$, $(\epsilon^{\zeta_1})^{\zeta_2}=\epsilon^{\zeta_1\zeta_2}$, so that Γ is a ring of operators for the group H of 1-units of K. A system of 1-units $\epsilon_1, \cdots, \epsilon_r$, together with, possibly, a p^sth root of unity η, is said to form a normal basis of H provided all elements of H are obtained in the form $\alpha=\eta^a\epsilon_1^{\zeta_1} \cdots \epsilon_r^{\zeta_r}$, and $\alpha=1$ implies $a \equiv 0 \pmod{p^s}$, $\zeta_1=\cdots=\zeta_r=0$. Krasner [Acta Arith. **3**, 133–173 (1939)] proved the theorems: (I) If K/k is of degree prime to p, then H has a normal basis. (II) Let K/k contain no primitive pth root of unity. Then, if K/k has no higher ramification, H has a normal basis. The author gives a proof of (I) employing the logarithm and $\mathfrak{P}$-adic exponential functions and based on his theorem: Let k be any $\mathfrak{p}$-adic number field, $\mathfrak{o}$ its ring of integers, G any group of order prime to p. Then the group ring of G over $\mathfrak{o}$ is a principal ideal ring. He shows by counterexamples that the hypothesis of (I) concerning the degree of K/k is indispensable, and proves the converse of (II).

R. Hull (Vancouver, B. C.).

Maass, Hans. Modulformen und quadratische Formen über dem quadratischen Zahlkörper $R(\sqrt{5})$. Math. Ann. **118**, 65–84 (1941). [MF 6310]

Notation: M a modular group in $R(\sqrt{5})$, that is, the system of matrices

$$S=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with integral $\alpha, \beta, \gamma, \delta$ in $R(\sqrt{5})$ and $\alpha\delta-\beta\gamma=1$; U a subgroup of M, in particular T the subgroup defined by the condition $\beta \equiv \gamma \equiv 0 \pmod 2$ or $\alpha \equiv \delta \equiv 0 \pmod 2$; a function $\phi(\tau, \tau')$ of two independent complex variables τ, τ', not identically vanishing, is called a modular form with the group U, the dimension $-r$ and the multiplier system $v(S)$ if it is regular in a fundamental domain of U and satisfies for all S in U the equation

$$\phi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}, \frac{\alpha'\tau'+\beta'}{\gamma'\tau'+\delta'}\right)=v(S)(\gamma\tau+\delta)^r(\gamma'\tau'+\delta')^r\phi(\tau, \tau'),$$

where $\alpha', \beta', \gamma', \delta'$ are the conjugates of $\alpha, \beta, \gamma, \delta$; let $a(r)$ be the number of linearly independent modular forms with the group M and the dimension $-r$.

The first result is concerned with the possible values of r and $v(S)$. In the case $U=M$, the dimension is necessarily a rational integer and $v(S)=1$; in the case $U=T$, the number $2r$ is a rational integer and there are exactly 4 possible systems of multipliers $v(S)$. In a former publication [Math. Ann. **117**, 538–578 (1940); cf. these Rev. **2**, 87] the author had proved the inequality $a(r)<cr^2$, with constant c. Now he uses results of F. Götzky [Math. Ann. **100**, 411–437 (1928)] and demonstrates that $a(1)=0$, $a(2)=1$, $a(3) \leqq 2$, $a(4) \leqq 3$. This leads to the three identities: $G_2^2=G_4$, $G_2=\theta_4$, $G_4=\theta_8$, where G_r $(r=2, 4)$ denotes the Eisenstein series of M with the dimension $-r$ whose Fourier expansion at the point ∞ has the constant term 1; moreover,

$$\theta_{2r}=\sum \exp(\pi i \operatorname{tr} Q(x_1, \cdots, x_{2r})\tau/\sqrt{5}),$$

where $Q(x_1, \cdots, x_{2r})=Q_{2r}$ is a certain positive even quadratic form in $R(\sqrt{5})$ with determinant 1 and $2r$ variables, with integral rational coefficients in the case $r=4$, and the summation is carried over all systems of integers $x_1, \cdots, x_{2r}$ in $R(\sqrt{5})$. It follows that for every integer in $R(\sqrt{5})$ the number of representations by Q_8 is the same as the number of representations by $Q_4(x_1, \cdots, x_4)+Q_4(x_5, \cdots, x_8)$, whereas those two quadratic forms are not equivalent in $R(\sqrt{5})$. There exist in $R(\sqrt{5})$ exactly two classes of positive even quadratic forms with determinant 1 and 8 variables, and one such class with 4 variables. *C. L. Siegel.*

Lehmer, D. H. Properties of the coefficients of the modular invariant $J(\tau)$. Amer. J. Math. **64**, 488–502 (1942). [MF 6944]

The author considers the coefficients $c(k)$ in the expansion $j(\tau)=1728J(\tau)=\sum_{k=-1}^{\infty}c(k)x^k$, where $x=e^{2\pi i\tau}$, and finds formulas by means of which $c(k)$ may be found recursively. One of these formulas determines $c(k)$ in terms of earlier c's, $\sigma_{11}(n)$ and $\tau(n)$, where $\sigma_{11}(n)$ is the sum of the eleventh powers of the divisors of n and $\tau(n)$ is Ramanujan's function defined by $\sum_{k=1}^{\infty}\tau(k)x^k=x\{(1-x)(1-x^2)(1-x^3) \cdots\}^{24}$. Multiplication formulas for the $c(k)$ are found, expressing $c(nk)$ in terms of earlier c's, but they are somewhat more complicated than the multiplication property enjoyed by $\tau(k)$. These formulas are useful for the computation of $c(k)$ when k is composite. Some congruence properties of the $c(k)$ are found and the author also discusses the coefficients $c_r(k)$ of $\{j(\tau)\}^r$. The methods of proof are connected with the transformation equation of the nth order for $j(\tau)$ and certain functional relations involving $j(\tau)$ are also found.

H. S. Zuckerman (Seattle, Wash.).

Weyl, Hermann. Theory of reduction for arithmetical equivalence. II. Trans. Amer. Math. Soc. **51**, 203–231 (1942). [MF 6315]

[The first part appeared in the same Trans. **48**, 126–164 (1940); cf. these Rev. **2**, 35.] Let $\mathfrak{F}$ be a field of finite degree f over the field of rational numbers, with the basis $\sigma_1, \cdots, \sigma_f$, and $[\mathfrak{o}]$ an order in $\mathfrak{F}$. Any n-uple $(\xi_1, \cdots, \xi_n)$ of numbers ξ_k $(k=1, \cdots, n)$ in $\mathfrak{F}$ is a vector. A set of vectors

constitutes a lattice L belonging to the order $[\mathfrak{F}]$ if $\mathfrak{x}+\mathfrak{y}$ and $\lambda\mathfrak{x}$ are contained in L for all $\mathfrak{x}, \mathfrak{y}$ in L and all λ in $[\mathfrak{F}]$. We assume that L is discrete and contains n linearly independent vectors. The unit lattice $I=I^{(n)}$ consists of all vectors $(\xi_1, \cdots, \xi_n)$ with ξ_k $(k=1, \cdots, n)$ in $[\mathfrak{F}]$. Two lattices L and Λ are called equivalent if one is carried into the other by a nonsingular linear transformation $\xi_k=\sum_{i=1}^{n}\delta_{ki}\eta_i$ $(k=1, \cdots, n)$; all linear transformations with $L=\Lambda$ form the modular group. If Λ contains I, then $j=[\Lambda : I]$ denotes the number of vectors in Λ which are incongruent modulo I; more generally, $j_k=[\Lambda^{(k)} : I^{(k)}]$, where $\Lambda^{(k)}$ consists of all vectors $(\xi_1, \cdots, \xi_k, 0, \cdots, 0)$ in Λ.

For arbitrary real variables $t_1, \cdots, t_f$, let $\tau^*=t_1\sigma_1+\cdots+t_f\sigma_f$, $\tau^{(l)}=t_1\sigma_1^{(l)}+\cdots+t_f\sigma_f^{(l)}$, where $\sigma_k^{(l)}$ denotes the lth conjugate of σ_k, and $|\tau^*|=\max|\tau^{(l)}|$. The f numbers $\tau^{(l)}$ (for the r real conjugates, $l=1, \cdots, r$) and $\frac{1}{2}(\tau^{(l)}+\tau^{(l+s)})$, $(1/2i)(\tau^{(l)}-\tau^{(l+s)})$ (for the s pairs of conjugate complex conjugates, $l=r+1, \cdots, r+s$; $r+2s=f$) are called the splitting coordinates of τ^*. Let $\xi_k^*=x_{k1}\sigma_1+\cdots+x_{kf}\sigma_f$, $\mathfrak{x}^*=(\xi_1^*, \cdots, \xi_n^*)$. A continuous function $F(\mathfrak{x}^*)=F(\xi_1^*, \cdots, \xi_n^*)$ of the nf real variables x_{kl} $(k=1, \cdots, n; l=1, \cdots, f)$ is called a gauge function if (1) $F(\mathfrak{x}^*)>0$ except for $\mathfrak{x}^*=0$, (2) $F(\tau^*\mathfrak{x}^*)\leqq|\tau^*|F(\mathfrak{x}^*)$, (3) $F(\mathfrak{x}_1^*+\mathfrak{x}_2^*)\leqq F(\mathfrak{x}_1^*)+F(\mathfrak{x}_2^*)$. There exist n linearly independent vectors $\theta_1, \cdots, \theta_n$ in L such that $F(\mathfrak{x})\geqq F(\theta_k)=M_k$ $(k=1, \cdots, n)$ for every vector $\mathfrak{x}$ in L outside $[\theta_1, \cdots, \theta_{k-1}]$. The mapping $\mathfrak{x}^*=\eta_1^*\theta_1+\cdots+\eta_n^*\theta_n\rightarrow(\eta_1^*, \cdots, \eta_n^*)$ carries $F(\mathfrak{x}^*)$ into a function $g(\eta_1^*, \cdots, \eta_n^*)$ and L into an equivalent lattice Λ which contains the unit lattice I for $[\mathfrak{F}]$. Let V_k be the volume defined by $g(\mathfrak{x}^*)<1$, $\xi_{k+1}^*=\cdots=\xi_n^*=0$, computed in terms of the splitting coordinates of $\xi_1^*, \cdots, \xi_k^*$, and let Δ denote the absolute value of the determinant of the $\sigma_k^{(l)}$ $(k, l=1, \cdots, f)$; then $(M_1\cdots M_k)^fv_kj_k\leqq(2^f\Delta)^k$ and $j_k\leqq(kf)!(4/\pi)^{kr}(\Delta/f!)^k$ $(k=1, \cdots, n)$. The main theorem of Minkowski's "Geometry of Numbers" is contained in these two inequalities, for the special case $f=1$.

The Hermitian forms $\gamma^*(\mathfrak{x}^*)=\sum_{k,l=1}^{n}\xi_k^*\gamma_{kl}^*\xi_l^*$ constitute a linear space of $\frac{1}{2}n(n+1)f-ns$ dimensions and the positive ones a convex cone G in that space. The two inequalities are applied to the gauge function F introduced by $F^2=\mathrm{tr}\,(\gamma^*(\mathfrak{x}^*))$; they lead to the construction of a fundamental domain in G for the modular group of an arbitrary lattice L belonging to an arbitrary order $[\mathfrak{F}]$. For $f=1$, this contains the result of Minkowski [J. Reine Angew. Math. **129**, 220–274 (1905)], and for $L=I$, $[\mathfrak{F}]=$principal order, the result of P. Humbert [Comment Math. Helv. **12**, 263–306 (1940); cf. these Rev. **2**, 148].

The whole theory goes through also in the case of a quaternion algebra $\mathfrak{F}$ over a totally real field if the norm is totally positive; the results are analogous to those in the case of a field $\mathfrak{F}$. *C. L. Siegel* (Princeton, N. J.).

Weyl, Hermann. On geometry of numbers. Proc. London Math. Soc. (2) **47**, 268–289 (1942). [MF 6596]

Let $f(\mathfrak{x})$ be any continuous function in the n-dimensional space of vectors $\mathfrak{x}=(x_1, \cdots, x_n)$ enjoying the following properties: $f(\mathfrak{x})>0$ except for the origin $\mathfrak{x}=\mathfrak{o}=(0, \cdots, 0)$; $f(t\mathfrak{x})=|t|f(\mathfrak{x})$ for any real factor t; $f(\mathfrak{x}+\mathfrak{x}')\leqq f(\mathfrak{x})+f(\mathfrak{x}')$. Let $2M_1$ be the minimum of $f(\mathfrak{a})$ for all lattice points $\mathfrak{a}\neq\mathfrak{o}$ and $\mathfrak{b}_1$ the corresponding value of $\mathfrak{a}$; let $2M_k$ $(k=2, \cdots, n)$ be the minimum of $f(\mathfrak{a})$ for all lattice points outside the $(k-1)$-dimensional linear manifold spanned by $\mathfrak{b}_1, \cdots, \mathfrak{b}_{k-1}$ and $\mathfrak{b}_k$ the corresponding value of $\mathfrak{a}$. Minkowski [Geometrie der Zahlen, Teubner, Leipzig, 1910, pp. 211–218] proved the inequality $M_1\cdots M_nV\leqq1$, where V denotes the volume of the convex solid $\mathfrak{K}$ defined by $f(\mathfrak{x})<1$. In the first part of his paper, the author derives a more general theorem concerning functions $\phi(\mathfrak{x})$ which satisfy the following 3 conditions: $0\leqq\phi(\mathfrak{x})\leqq1$; $\phi(\mathfrak{x})=0$ outside a finite region; the locus of the points $\mathfrak{x}$ where $\phi(\mathfrak{x})\geqq r$ is a convex set for any $r>0$. Let

$$\phi^{(k)}(\mathfrak{x})=\phi(\mathfrak{x})\int_0^1\prod_{\mathfrak{a}}\{1-t\phi(\mathfrak{x}-\mathfrak{a})\}dt,$$

the product extending to all vectors $\mathfrak{a}\neq\mathfrak{o}$ of a k-dimensional discrete lattice $(1\leqq k\leqq n)$, and $\phi_q(\mathfrak{x})=\phi(q^{-1}\mathfrak{x})$ $(q>0)$,

$$J^{(k)}[\phi]=\int\phi^{(k)}(\mathfrak{x})dx_1\cdots dx_n,$$

where the integral extends over the whole $\mathfrak{x}$-space; then $q^{k-n}J^{(k)}[\phi_q]$ is a monotonically increasing function of q. This theorem leads to Minkowski's inequality if ϕ is chosen as the characteristic function of the convex solid $\mathfrak{K}$, that is, $\phi=1$ inside $\mathfrak{K}$ and $\phi=0$ outside.

It seems that Minkowski did not notice a simple idea connecting his inequality with the problem of reduction of the lattice in terms of the given convex solid $\mathfrak{K}$. This idea was found by K. Mahler [Quart. J. Math., Oxford Ser. **9**, 259–262 (1938)]. In the second part, the author obtains independently again Mahler's result and applies it in particular to the reduction of quadratic forms. *C. L. Siegel.*

Hlawka, Edmund. Über komplexe homogene Linearformen. Monatsh. Math. Phys. **49**, 321–326 (1941). [MF 6855]

This paper gives a new proof of a theorem of Minkowski which may be stated as follows: Let $\alpha, \beta, \gamma, \delta$ be any complex numbers for which $\alpha\delta-\beta\gamma=1$. Then there exist complex integers x, y, not both zero, such that $\alpha x+\beta y$ and $\gamma x+\delta y$ simultaneously do not exceed $\rho=((3^{\frac{1}{2}}+1)/6^{\frac{1}{2}})^{\frac{1}{2}}=1.05610\cdots$ in absolute value. The author uses the two Hermitian forms

$$|\alpha x+\beta y|^2\pm|\gamma x+\delta y|^2$$

and, denoting these by $F_1(x, y)$ and $F_2(x, y)$, shows the above theorem is equivalent to the assertion of the existence of x and y such that

$$f(x, y)=F_1(x, y)+|F_2(x, y)|\leqq2\rho^2.$$

By supposing that $f(0, 1)>2\rho^2$ it is shown that $f(-1, 1)\leqq2\rho^2$. *D. H. Lehmer* (Berkeley, Calif.).

Jarník, Vojtěch. Zur Gitterpunktlehre der Ellipsoide $\alpha_1(u_1^2+\cdots+u_{r_1}^2)+\alpha_2(u_{r_1+1}^2+\cdots+u_r^2)\leqq x$. II. Časopis Pěst. Mat. Fys. **70**, 1–33 (1940). (German. Czech summary) [MF 5426]

Let

$$Q(u)=\alpha_1(u_1^2+u_2^2+\cdots+u_{r_1}^2)+\alpha_2(u_{r_1+1}^2+\cdots+u_r^2),$$

$\alpha_1>0$, $\alpha_2>0$, α_1/α_2 irrational, $s=\min(r_1, r_2)\geqq4$ a positive definite quadratic form. Denote by $A(x)$ the number of lattice points in the ellipsoid $Q(u)\leqq x$, and by $V(x)$ the volume of this ellipsoid. Write $P(x)=A(x)-V(x)$. Let p_ν/q_ν be the convergents of α_1/α_2. Moreover, let

$$f(Q)=\overline{\lim}\,(\log|P(x)|/\log x),$$
$$\gamma(\alpha_1, \alpha_2)=\overline{\lim}\,(\log q_{\nu+1}/\log q_\nu).$$

It was known that (proved by the author in previous papers) $\lim(P(x)/x^{s/2-1})=0$ and $f(Q)=r/2-1-(\gamma(\alpha_1, \alpha_2))^{-1}$. The author proves among others the following results: Assume that $q_{\nu+1}/q_\nu$ is bounded; then for $s>6$, $\overline{\lim}\,(P(x)/x^{s/2-\varepsilon})<\infty$.

Let $\varepsilon>10$; then, for every $x>c_6$, there exist two numbers x_1 and x_2 such that $x-c_6/x<x_j<x$ $(j=1, 2)$, $P(x_1)>c_7x_1^{r/2-2}$, $P(x_2)<-c_7x_2^{r/2-2}$. On the other hand, if $x_2>x_1>1$ and $P(x_1)>\varepsilon x_1^{r/2-2}$, $P(x_2)<-\varepsilon x_2^{r/2-2}$, then $x_2-x_1>c_8\varepsilon/x_2$.

P. Erdös (Philadelphia, Pa.).

Jarník, Vojtěch. Über die Mittelwertsätze der Gitterpunktlehre. V. Časopis Pěst. Mat. Fys. **69**, 148–174 (1940). (German. Czech summary) [MF 5207]

Let $Q(x)=\sum_{u,v=1}^{r}c_{u,v}x_ux_v$ be a positive definite form with determinant D. Denote by $A(y)$ the number of lattice points in the ellipsoid $Q(x)\leqq y$. Write

$$P(y)=A(y)-\pi^{r/2}y^{r/2}/D^{\frac{1}{2}}\Gamma(r/2-1), \quad M(y)=\int_0^y P^2(y)dy.$$

The author proves that for $r>2$, if the c's are integers, there exists an $H>0$ (depending only on Q) such that $M(x)=Hx^2\log x-O(x^2\log^{\frac{1}{2}}x)$ for $r=3$ and $M(x)=Hx^{r-1}-O(g(x))$ for $r>3$, where $g(x)=x^{5/3}\log x$ for $r=4$, $g(x)=x^3\log^2 x$ for $r=5$ and $g(x)=x^{r-2}$ for $r>5$. He also shows that the error term is best possible for $r>5$. [This is not difficult.] *P. Erdös* (Philadelphia, Pa.).

Shatrowsky, L. On two Erdös' theorems for lattice point sets of the space of n dimensions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 411–422 (1941). (Russian. English summary) [MF 6830]

In this paper theorems of Erdös [Acta Arith. **1**, 197–200 (1936); Trav. Inst. Math. Tbilissi **3**, 217–233 (1938)] and Landau [Über einige neuere Fortschritte in der additiven Zahlentheorie, Cambridge Tract no. 35, Cambridge University Press, Cambridge, England, 1937, pp. 60–62] are partially generalized. The generalization consists in considering n-dimensional vectors with nonnegative integral components instead of nonnegative integers. *M. Kac.*

Vijayaraghavan, T. On the fractional parts of the powers of a number. II. Proc. Cambridge Philos. Soc. **37**, 349–357 (1941). [MF 5543]

Let $G(\theta)$ denote the limit points of the fractional parts of θ^n $(n=1, 2, \cdots)$. Let S denote the set of all algebraic integers θ whose conjugates $\theta, \theta_1, \cdots, \theta_k$ satisfy $|\theta_i|<1$ $(i=1, \cdots, k)$. Since $\theta^n+\sum\theta_i^n$ is an integer for every n, $G(\theta)$ has at most the two points 0 and 1; if θ is irrational the point 1 must belong to $G(\theta)$. The set S contains algebraic integers of all degrees. However, there exist algebraic integers for which $G(\theta)$ consists of the entire interval $(0, 1)$. The principal result is: if θ is a real algebraic number not less than 1, and if $G(\theta)$ consists of a finite number of elements, then θ is an algebraic integer, and if θ is irrational, θ belongs to S. More generally, if the limit points of the fractional parts of $u_n=d_1\xi_1^n+\cdots+d_l\xi_l^n$ $(n=1, 2, 3, \cdots)$ are finite in number, the d_j being nonzero constants and the ξ_j distinct algebraic numbers numerically greater than 1, then the ξ_j are algebraic integers, and any conjugates of any of the ξ_j not among them are numerically less than 1. The proof uses a theorem of Fatou and Hurwitz [Math. Ann. **77**, 510–512 (1916)] on recurring sequences of rational integers, and several deep-lying results from algebra and analysis. An extension to nonreal algebraic numbers is indicated at the end. *G. Pall* (Montreal, Que.).

Rosser, Barkley. A generalization of the Euclidean algorithm to several dimensions. Duke Math. J. **9**, 59–95 (1942). [MF 6342]

The author considers a generalization of the Euclidean algorithm to the case of vectors in 2, 3 and 4 dimensions and discusses in detail several applications of the algorithm as mentioned below. The one dimensional case of two collinear vectors is practically what is found in Euclid's Elements, with the exception that the author prefers to use the "absolutely least remainder" algorithm. The greatest common divisor (if it exists) of more than two vectors $V_1, V_2, \cdots, V_n$ is defined as a set of vectors $U_1, U_2, \cdots, U_n$ with the following three properties: (a) the U's are linearly independent (with respect to integer coefficients); (b) each U is a linear (integer) combination of the V's; (c) each V is such a combination of the U's. If the set of all linear combinations of the V's has no limit point the V's are said to be commensurable, otherwise incommensurable. The 2-dimensional generalization of the Euclidean algorithm starts with three commensurable coplanar vectors, say U_1, V_1, W_1, arranged according to increasing length. Since these vectors are coplanar $W_1=xU_1+yV_1$. The vector $X=W_1-m_1U_1-n_1V_1$, where m_1, n_1 are integers nearest to x and y, is shorter than V_1. We now take U_2 to be the shorter of U_1 and X, and take V_2 to be the longer. Finally W_2 is defined as V_1. Then the sum of the lengths of U_2, V_2, W_2 is less than that of U_1, V_1, W_1. Repeating the process we get a sequence of triplets (U_n, V_n, W_n) the sum of whose lengths is a decreasing function of n. Since U_1, V_1, W_1 are commensurable, no sequence of linear combinations of them has a limit point so the process cannot go on indefinitely but breaks down by U_n and V_n being collinear for some n. This gives (W_n, G), where G is the greatest common divisor of U_n and V_n, as the greatest common divisor of U_1, V_1, W_1. In case U_1, V_1, W_1 are incommensurable, the process continues indefinitely giving an infinite sequence of partial quotient pairs (m_i, n_i).

For the d-dimensional generalization with $d>2$, we start with $d+1$ dependent vectors $U_1^{(\nu)}$ $(\nu=0, 1, \cdots, d)$ arranged according to increasing length and choose the vector X as the shortest of all vectors of the form

$$U_1^{(d)}-\sum_{i=0}^{(d-1)}m_iU_1^{(i)},$$

where m_i are integers. A constructive method for doing this is explained. For $d=3$ and 4 it is proved that X is shorter than $U_1^{(d-1)}$. That this fact is false for $d=5$ is shown by an example. Hence the straightforward generalization of the above algorithm will not work for $d\geqq 5$.

The algorithm is applied to the six following problems: (I) To find the minimum (nonzero) value of a positive definite quadratic form in four variables. (II) Given four linear forms in four variables, to find a set of values of the variables which make each form less than the fourth root of their determinant, in absolute value. (III) To find a set of n fractions, all with the same denominator D, which approximate n given real numbers to within an error less than $D^{-1/n}$ in absolute value. (IV) Let $L=\sum_{i=1}^{n}A_ix_i$ be a given linear form in n variables with integer coefficients A_i. To find a general solution of $L=k_1$ in the form $x_i=\sum_{j=1}^{n}a_{ij}k_j$, where the k's are arbitrary integers, for which the sum of the squares of the a_{ij} is a minimum. (V) Let $L(x, y, z)=0$ be the equation of a plane P in 3-space. To find that lattice point inside the sphere $x^2+y^2+z^2=R^2$ lying nearest to P. (VI) Given two real symmetric positive definite matrices A and B to discover whether they are equivalent and, if so, to exhibit all matrices C such that $B=CAC^T$ (the latter matrix being the transpose of C). All problems are illustrated by one or more numerical examples. Problem (III)

is solved by a process which seems to work in every case although the author is unable to prove this.

D. H. Lehmer (Berkeley, Calif.).

Yannopoulos, Constantin. Zur Kettenbruchtheorie im Dreidimensionalen. Math. Z. **47**, 105–110 (1940). [MF 6301]

G. Bullig, in connection with his algorithm for determination of extreme parallelepipeds in three dimensions [Abh. Math. Sem. Hansischen Univ. **13**, 321–343 (1940); these Rev. **2**, 253; Mitt. Math. Ges. Hamburg **8**, part 2, 164–187 (1940); these Rev. **2**, 351], has constructed a graph consisting of cycles

$$Q=\Omega_{23}\Omega_{21}^{m_2}\Omega_{31}\Omega_{23}^{m_3}\Omega_{13}\Omega_{12}^{m_1}(Q),$$

where m_i $(i=1, 2, 3)$ are nonnegative integers, Q is a node corresponding to an extreme parallelepiped and Ω_{kj} $(k=1, 2, 3; j=1, 2, 3; k\neq j)$ are the processes defined in his paper Z1. For the special case of a number lattice the author here gives a plane construction of the graph in which the cycles are represented by four kinds of convex polygons.

D. C. Spencer (Stanford University, Calif.).

NUMERICAL AND GRAPHICAL METHODS

***Table of Natural Logarithms. Vol. IV.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 65-2-97-33; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1941. xxii+506 pp. $2.00.

For the first two volumes, see these Rev. **2**, 366; for the third volume, see these Rev. **3**, 152. The present concluding volume contains the natural logarithms, to sixteen decimal places, of the numbers from 5 to 10 in steps of .0001. Moreover there are listed the natural logarithms of the first ten integers to 40 decimal places, and the values, to 25 decimal places, of $\log_e(1+x)$ and $-\log_e(1-x)$, where x is of the form $k\ 10^{-n}$ with $k=1, \cdots, 9$ and $n=1, 2, \cdots, 13$. With the aid of these tables it is possible to obtain the logarithm of any number to 24 decimal places.

***Natural Sines and Cosines to Eight Decimal Places.** U. S. Dept. Commerce, Coast and Geodetic Survey, Special Publ. no. 231. United States Government Printing Office, Washington, D. C., 1942. 541 pp. $1.75.

Tables are in steps of 1″.

Buerger, M. J. Numerical structure factor tables. Geolog. Soc. America. Special Papers no. **33**, 119 pp. (1941). [MF 6915]

A method is described for a simple computation of the x-ray diffraction intensities expected from a given crystal structure. All but the first eleven pages of the book consist of tables to facilitate this computation. These include: (i) three place double entry tables for $\cos 2\pi hx$ and $\sin 2\pi hx$, for $h=1, 2, \cdots, 30$ and values of x between 0 and 1 in steps of .001; (ii) four place tables for $z=(1+\cos^2 2\theta)/\sin 2\theta$ as a function of $\sin\theta$ in steps of .001; (iii) similar tables for $z^{\frac{1}{2}}$, z^{-1} and $z^{-\frac{1}{2}}$.

***Tables of the Moment of Inertia and Section Modulus of Ordinary Angles, Channels, and Bulb Angles with Certain Plate Combinations.** Prepared by the Federal Works Agency, Work Projects Administration for the City of New York, as a Report of Official Project No. 165-2-97-22, Mathematical Tables Project; conducted under the sponsorship of the National Bureau of Standards. Technical Director: Arnold N. Lowan. New York, 1941. xiii+197 pp. $1.25.

Sherman, J. and Ewell, R. B. A six-place table of the Einstein functions. J. Phys. Chem. **46**, 641–662 (1942). [MF 6961]

Six-place tables are given for the values of $x/(e^x-1)$, $x^2e^x/(e^x-1)^2$ and $-\log(1-e^{-x})$. The values of x are 0(.005)3(.01)8(.05)15.

Uhler, Horace S. The coefficients of Stirling's series for $\log\Gamma(x)$. Proc. Nat. Acad. Sci. U. S. A. **28**, 59–62 (1942). [MF 6138]

A table of the first 71 coefficients c_n in Stirling's asymptotic formula for $\log\Gamma(x)$ is given to a steadily decreasing number of significant figures, starting with 121 for c_9, the first coefficient for which a complete period was not obtained, and ending with 2 for c_{71}. Values of 100! and of $\log_e(100!)$ to 156 places are also given. *P. W. Ketchum.*

Orr, William J. C. Expansions for a particular class of exponential-logarithmic integrals. Proc. Cambridge Philos. Soc. **38**, 34–39 (1942). [MF 6033]

The functions studied are

$$F(\alpha, s, n)=\int_0^\infty y^s[\log(1+y)]^n e^{-\alpha y^2}dy.$$

Values for any s can be found by successive differentiation of $F(\alpha, 0, n)$ and $F(\alpha, 1, n)$. Recurrence relations on n involving these latter functions and their derivatives are found. Starting with the known values of $F(\alpha, 0, 0)$ and $F(\alpha, 1, 0)$, these differential equations may be integrated step by step, yielding a sum of power series in α. Tables of the first 12 coefficients of these series are given for $n=1, 2, 3$. Six place tables of $F(\alpha, 0, n)$ and $F(\alpha, 1, n)$ are given for $n=1, 2, 3$ and $\alpha=0.1(0.1)1.5$. *P. W. Ketchum.*

Rosenberg, Paul. Evaluation of functions related to Tait's mean free path. Phys. Rev. (2) **61**, 528–530 (1942). [MF 6472]

A table of values, to seven significant figures, of the functions

$$\Psi(x)=xe^{-x^2}+(2x^2+1)\int_0^x e^{-y^2}dy$$

and $x^3/\Psi(x)$ is given over the range 0(.05)1.5(.1)10(.2)20.4. There is also a table, to seven significant figures, of

$$\Phi(x)=x^2\int_0^\infty y^3e^{-xy^2}dy/\Psi(y)$$

over the range for 28 different values of x between 0 and 40. The tables are used to recalculate the values of the constants

$$\theta_n=\int_0^\infty 4x^ne^{-x^2}dx/\Psi(x), \qquad n=3, 4, 5,$$

which occur frequently in kinetic theory. *W. Feller.*

Dwight, Herbert B. A five-figure table of the Bessel function $I_n(x)$. Trans. Amer. Inst. Elec. Engrs. **60**, 135–136 (1941).

This is a table of the modified Bessel function of the first kind $I_n(x)=i^{-n}J_n(ix)$ for $n=0, 1, 2, \cdots, 10, 11$ and x ranging from 0 to 6 at intervals of 0.1. *I. Opatowski.*

Lancaster, Otis E. Machine method for the extraction of cube root. J. Amer. Statist. Assoc. **37**, 112–115 (1942). [MF 6352]

The author shows that the sequence of numbers defined by

$$2a_{n+1}=(a_n^3+2A)(a_n^3+\tfrac{1}{2}A)^{-1}a_n$$

tends to the cube root of A more rapidly than the commonly used approximations defined by $3a_{n+1}=2a_n+Aa_n^{-2}$. A procedure for use on a calculating machine is outlined. *W. Feller* (Providence, R. I.).

Kerawala, S. M. A rapid method for calculating the least squares solution of a polynomial of degree not exceeding the fifth. Indian J. Phys. **15**, 241–276 (1941). [MF 6249]

The variable y, a function of x that is known for each of n equally spaced values of x, is to be represented by a polynomial, of degree ρ in x, with coefficients such as to minimize the sum of the squares of the residuals in y. The x's are transformed linearly so as to yield a set of x''s, spaced symmetrically about zero and at unit intervals. The coefficients in the least squares polynomial in x' are then linear functions of the y's. The paper tabulates the coefficients in the linear functions, for all positive values of n less than 31 and of ρ less than 6. The polynomial in x' is finally transformed to a polynomial in x. *T. E. Sterne.*

Schmidt, R. J. On the numerical solution of linear simultaneous equations by an iterative method. Philos. Mag. (7) **32**, 369–383 (1941). [MF 6042]

Let the given equations be $\sum_{k=1}^{n}a_{ik}x_k=b_i$ $(i=1, \cdots, n)$. Starting from arbitrary approximations $x_2^{(0)}, \cdots, x_n^{(0)}$, we use the first equation to find an approximation $x_1^{(1)}$ to x_1. Using the values $x_1^{(1)}, x_3^{(0)}, \cdots, x_n^{(0)}$, we find, from the second equation, a new approximation $x_2^{(1)}$ to x_2, etc. The values $x_i^{(r)}$ thus found are called the rth iterates. Now let $p_n\lambda^n+p_{n-1}\lambda^{n-1}+\cdots+p_1\lambda=0$ be the expansion of the equation

$$\begin{vmatrix} \lambda a_{11} & a_{12} & \cdots & a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nn} \end{vmatrix}=0.$$

The author shows that, if $p_n+p_{n-1}+\cdots+p_1=0$, the original equations are either inconsistent or not independent. Otherwise, the true values of the unknowns are given by

$$x_i=\frac{p_nx_i^{(n-1)}+p_{n-1}x_i^{(n-2)}+\cdots+p_2x_i^{(1)}+p_1x_i^{(0)}}{p_n+p_{n-1}+\cdots+p_2+p_1}.$$

Thus, theoretically, the unknowns can always be found by this iterative method. The author outlines several simplified methods of finding the unknowns approximately. For details we must refer to the paper. *W. Feller.*

Dwyer, Paul S. The Doolittle technique. Ann. Math. Statistics **12**, 449–458 (1941). [MF 6054]

This paper offers a formal proof of the validity of the Doolittle technique for the solution of numerical simultaneous linear equations. The author also points out advantages of his "abbreviated" Doolittle method. *A. L. Foster.*

Danielson, G. C. and Lanczos, C. Some improvements in practical Fourier analysis and their application to X-ray scattering from liquids. J. Franklin Inst. **233**, 365–380, 435–452 (1942). [MF 6378]

The authors are concerned with Fourier interpolation with an even number of equidistant points. They devise a practical scheme intended to reduce the computational work; they show that a simple matrix scheme can be used in place of available forms. It is shown furthermore that using intermediate ordinates it becomes possible to estimate, before calculating any coefficients, the probable accuracy of the analysis (and hence the number of coefficients to be calculated). Having obtained the coefficients they show how to calculate any intermediate value of the Fourier integral by interpolation. Numerical examples (X-ray scattering) are given. [In connection with computational schemes K. Stumpf: Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung, Springer, Berlin, 1939 might be mentioned.] *W. Feller* (Providence, R. I.).

Berkeley, Edmund C. Summation as a function of any terms. Record **29**, 314–348 (1940). [MF 6914]

The author is concerned with formulas of Lubbock's type for the evaluation of finite sums. The formula is, however, refined by an adaptation of Gauss's procedure for numerical integration. Thus, for given n and k, the author finds the optimum position of the k "anchor terms" to be used for the computation of a sum of n terms. For $n\leqq100$ and $k\leqq5$ extensive tables are given showing the subscripts of the k anchor terms together with the corresponding coefficients. *W. Feller* (Providence, R. I.).

Lal, D. N. and Dasgupta, P. N. Interpolation polynomials in two and more variables. Bull. Calcutta Math. Soc. **32**, 7–14 (1940). [MF 6152]

Aitken's method of interpolation by linear and quadratic cross-means is generalized to functions of two variables. The authors have apparently overlooked the fact that this was done by Aitken himself in his original paper quoted by them [Proc. Edinburgh Math. Soc. (2) **3**, 56–76 (1932), in particular, § 6]. From a practical viewpoint Aitken's arrangement seems preferable. *W. Feller.*

Das Gupta, P. N. On an interpolation formula connected with a definite integral in n-variables. Bull. Calcutta Math. Soc. **33**, 41–44 (1941). [MF 6167]

Burnside's quadrature formula for $\int_{-1}^{1}\int_{-1}^{1}f(x, y)dxdy$, where $f(x, y)$ is a polynomial of degree not exceeding five, is generalized to polynomials of any degree in n variables. *W. Feller* (Providence, R. I.).

Majid Mian, A. and Chapman, S. Approximate formulae for functions expressed as definite integrals. Philos. Mag. (7) **33**, 115–130 (1942). [MF 6257]

The main interest of the paper centers around the integrals

$$E_n(t)=\int_1^\infty e^{-tx}x^{-n}dx, \quad B_n(t)=2\pi^{-1}\int_0^{\pi/2}e^{-t\sec x}\cos^n x\,dx.$$

Various expressions for, and approximations to, these integrals are available. However, according to the authors, they are inconvenient for moderate values of n and for, say, $1 \leqq t \leqq 5$. The authors find that, over a considerable range of n and t, the integrals can be approximated by "index-sums" of type $\sum a_k \lambda_k^{n-2} e^{-\lambda_k t}$; usually four or five terms suffice. It is shown that this method can be used in various other cases. No theoretical estimate of the error is given.
W. Feller (Providence, R. I.).

Sauer, R. und Pösch, H. Rechnerische Differentiation von Kurven. Z. Verein. Deutsch. Ingenieure **85**, 195–197 (1941). [MF 6958]

The paper compares the merits of the following formulas for the numerical calculation of derivatives:

(1) $y_0' = (2h)^{-1}(y_1 - y_{-1})$,

(2) $y_0' = (2h)^{-1}(\Delta y_0 + \Delta y_{-1}) + (5h)^{-1}(\Delta^3 y_{-1} + \Delta^3 y_{-2})$,

(3) $y_0' = (2h)^{-1}(\Delta y_0 + \Delta y_{-1}) - (12h)^{-1}(\Delta^3 y_{-1} + \Delta^3 y_{-2})$.

The first is familiar, while the second and third are obtained by least squares from five consecutive ordinates, (2) using a second degree polynomial and (3) a third degree polynomial. If the y's are affected by random errors the authors distinguish two cases, the "usual" case where second and third differences are of the same order of magnitude, and the "exceptional" case where third and fourth differences are of the same order of magnitude. Their conclusion is: in the usual case (2) is best; in the exceptional case either (1) or (3) is best. *W. E. Milne* (Corvallis, Ore.).

Bickley, W. G. and Miller, J. C. P. Numerical differentiation near the limits of a difference table. Philos. Mag. (7) **33**, 1–14 (1942). (4 plates) [MF 6226]

The familiar formulae for numerical differentiation are based on finite difference formulae containing only forward, backward or central differences, respectively. The order of the highest derivative which can be computed is that of the highest difference which can be used; near the limits of a difference table this order will be different at different sections. The author therefore develops "mixed" formulae commencing as central difference formulae and switching over to forward (or backward) differences when the edge of the difference table is reached; the route through the difference table consists of a section of a horizontal line and a section of a diagonal line. Tables are given for the coefficients in formulae for the derivatives at the tabulated values of the argument and for arguments at mid-interval. The arrangement is such that the coefficient to be used as multiplier occupies the same position as the difference which it is to multiply occupies in the difference scheme. With the mixed formulae the user can at any stage switch from central to forward or backward differences. The tables enable one to calculate the first four derivatives using mixed type formulae, and the first twelve derivatives using forward or central differences. The use of these extremely practical tables is illustrated by several examples. *W. Feller.*

Milne, W. E. The numerical integration of $y'' + g(x)y = f(x)$. Amer. Math. Monthly **49**, 96–98 (1942). [MF 6241]

The author applies the central difference operator δ^2 to the equation given in the title and to the well-known relation $h^2 y'' = \delta^2 y - \delta^4 y/12 + \cdots$. In this way he obtains four equations from which he eliminates the three quantities y'', $\delta^2 y''$, $\delta^4 y$. By an appropriate substitution the resulting equation is reduced to $\delta^2 z + G(x)z = F(x) - \delta^4 y/240 + \cdots$. Neglecting the higher differences reduces this equation to a simple linear difference equation of the second order.
W. Feller (Providence, R. I.).

***Grammel, R. Ein Beitrag zur Lösung des Dreikörperproblems.** Astronomical Papers dedicated to Elis Strömgren, pp. 40–50. Einar Munksgaard, Copenhagen, 1940.

In an earlier publication [Ing.-Arch. **10**, 395–411 (1939); these Rev. **1**, 253] the author developed a graphical method of differentiation and integration. In this article the construction is shown for a system of equations $p''(\theta) = F(p, q, p', q')$, $q''(\theta) = G(p, q, p', q')$, the values of p'', q'' being computed numerically from the graphically obtained values of the reciprocals of p, q, p', q'. The application to an orbit in the restricted problem of three bodies with equal masses of the two finite bodies is given as an illustration. [Note of the reviewer: The method is elegant and of theoretical interest; its practical value is limited by the fact that the derivatives of the highest order are computed numerically. This is the most laborious part in practically all problems to which the method of numerical integration is usually applied. Nothing seems to be gained by replacing the simple operations of numerical integration by graphical construction. The accumulation of errors is of the same nature in both methods.] *D. Brouwer* (New Haven, Conn.).

Morgans, W. R. On the solution of second order differential equations satisfying boundary conditions. Philos. Mag. (7) **32**, 483–488 (1941). [MF 6047]

The author considers the solution of the differential equation $d^2V/dx^2 + \kappa^2 V = 0$ over the interval $0 < x < 1$ along with the boundary conditions $V = 0$ for $x = 0, 1$. He obtains an approximate solution by breaking up the interval $(0, 1)$ into n equal parts, using a straight line approximation V_r to V in the rth interval such that the values of V_r join on continuously from one interval to the other but have discontinuous derivatives in accordance with $n[(dV_{r+1})/dx - (dV_r)/dx] + \kappa^2 V_r = 0$ at $x = r/n$. The latter condition is suggested by an analogy derived from potential theory, according to which V may be interpreted as the potential of a proper volume distribution of charge, while the approximation in question may be found as the potential due to surface distributions of charge from the intervals in question to their boundaries. It is also derivable from a parabolic approximation to V through three adjacent points of division. It is shown that after the boundary conditions are imposed at $x = 0$, $x = 1$ the approximate solution in question approaches an exact solution as n becomes infinite. Since both the exact solution and its present approximation with n intervals are available in finite form, the degree of approximation due to any n may also be determined. An analogous treatment of the same differential equation with the boundary conditions $V = 1$ when $x = 0, 1$ is given.

The two-dimensional differential equation

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \kappa^2 V = 0$$

vanishing along the boundary of the square $0 < x, y < 1$ is next considered by a similar method. This problem is treated by breaking up both the x and the y intervals into equal parts and approximating to the function V over each one of the resulting rectangles by means of a polynomial in x, y which is a linear combination of four functions $1, x, y, xy$, the constants of each polynomial being determined in such a way as to have the approximations join on continuously along the edges of adjacent rectangles, while possessing

properly discontinuous first derivatives similar to the one-dimensional case. For the relatively simple square-shaped two-dimensional region in question, the resulting approximation to V may be obtained as a product of a function of x by a function of y. The three-dimensional analogous problem is also treated. *H. Poritsky.*

Schwank, Friedrich. Die unmittelbaren Verfahren der Variationsrechnung als Hilfsmittel des Ingenieurs. Z. Verein. Deutsch. Ingenieure **84**, 313–317 (1940). [MF 6962]

An expository article describing the Rayleigh-Ritz method.

James, Hubert M. Some applications of the Rayleigh-Ritz method to the theory of the structure of matter. Bull. Amer. Math. Soc. **47**, 869–884 (1941). [MF 5932]

In this address, delivered before the American Mathematical Society, the physicist presents a review of the numerical applications of the Rayleigh-Ritz method to problems in atomic and molecular theory. The results thereby obtained by the application of enough hard work are mostly very satisfactory. The agreement with the observed value is practically within the accuracy of observation. In wave mechanics the wave functions ψ are determined as the properly continuous functions which extremalize the average energy $\bar{E}=\int\psi^*H\psi d\tau/\int\psi^*\psi d\tau$ (H represents the Hamiltonian operator). The function ψ depends, even in the simplest cases, on a great number of variables. The critical step in the application of the Ritz method is the choice of the set of coordinate functions in terms of which the wave functions are to be expressed as sums. The physicist knows, however, that in certain cases he needs to consider only functions symmetric under interchange of electrons. If the excited state is not the lowest, one does not obtain the energy of the state as the lowest root of a secular equation, but as a higher root. The approximate energy thus computed is always an upper limit for the energy of the corresponding state of the atom. [The reviewer would like to mention here the curious fact that Ritz himself did not use the secular equation in his computations of the higher frequencies of a free plate. In other words, Ritz did not use what is called to-day the Ritz method]. The address contains references to papers by E. A. Hylleraas, J. K. L. MacDonald, H. M. James, A. S. Coolidge, H. Bethe, G. Breit and others. *A. Weinstein.*

Donnell, Lloyd Hamilton. Some refinements in methods of graphical integration. J. Franklin Inst. **233**, 331–348 (1942). [MF 6377]

The paper is concerned with the graphical integration of first order differential equations. The author starts by constructing a chart of curves obtained by setting the angle of inclination, that is, arc tan y', equal to a number of equally spaced constants. This has the advantage over the usual method, where the slope is given several equally spaced values, that the average curvature of the solution between two intersections with neighboring curves of the chart is inversely proportional to the length between the intersections. This property is made use of in a graphical method of successive approximation proposed by the author. A drafting device for obtaining a variety of curves by means of a flexible strip with variable angular constraints at each end is also described. *P. W. Ketchum* (Urbana, Ill.).

Behrbohm, Hermann. Graphische Konstruktion von Sturzfluggeschwindigkeiten. Luftfahrtforschung **17**, 167–171 (1940). [MF 6560]

The author considers the differential equation of a vertical fall, assuming that the density of the air is given by $\rho=ae^{-bz}$ (where z is the height) and the air resistance is proportional to v^2. Since $v=-dz/dt$, we have $\dot{v}=-\frac{1}{2}d(v^2)/dz$, so that the given equation reduces to a linear equation in v^2. The author constructs simple nomograms for the solution which permit one to find the velocity at height z if the initial height and velocity are given. *W. Feller.*

Pekeris, C. L. and White, W. T. Differentiation with the cinema integraph. J. Franklin Inst. **234**, 17–29 (1942). [MF 6912]

It is noted that ordinary processes of numerical differentiation involve essentially the summation of neighboring ordinates to which weighting factors have been applied. The authors extend the method to the computation of continuous numerical derivatives. Polynomial approximation of the function to be differentiated is replaced by a development into a finite series of a set of orthogonal functions. By suitable choice of functions, the integrations required for evaluation of coefficients are put into the form

$$R(x)=\int_a^{b \text{ or } s} P(x\pm z)Q(z)dz.$$

This integral is of the general type which can be evaluated on the cinema integraph [Hazen and Brown, J. Franklin Inst. **230**, 19–44, 183–205 (1940); these Rev. **2**, 62]. *S. H. Caldwell* (Cambridge, Mass.).

Hedeman, Walter R., Jr. The cinema integraph in interreflection problems. J. Math. Phys. Mass. Inst. Tech. **20**, 402–417 (1941). [MF 6109]

The author describes the use of the cinema integraph, an optical calculating machine for evaluating an integral containing a parameter, in the solution of integral equations. It is illustrated by application to a Fredholm type linear integral equation of second kind arising in the interreflection problem of luminosity along the interior of a right circular cylinder of finite length. The reciprocal kernel is first found as the solution of an integral equation from which the luminosity function for any exciting luminosity can be found by explicit integration. A method of evaluating errors is given, also numerical results for a cylinder whose length equals its diameter. *H. L. Hazen.*

Varney, Robert N. An all electric integrator for solving differential equations. Rev. Sci. Instruments **13**, 10–16 (1942). [MF 6095]

The essential operation which must be performed in solving differential equations by means of mechanisms can be carried out by various devices which are available commercially. The author describes a number of such devices and illustrates the method for two specific examples. *S. H. Caldwell* (Cambridge, Mass.).

Beard, R. E. The construction of a small-scale differential analyser and its application to the calculation of actuarial functions. J. Inst. Actuar. **71**, 193–211; discussion, 212–227 (1942). (3 plates) [MF 6521]

An excellent description is given of the construction of a simplified differential analyzer built for the use of the author

in actuarial work. Representative examples of the application of machine processes to actuarial functions are given in detail. The discussion which follows the paper brings out a number of interesting historical details and emphasizes the need for mechanical aid in actuarial computations.

S. H. Caldwell (Cambridge, Mass.).

Shannon, Claude E. Mathematical theory of the differential analyzer. J. Math. Phys. Mass. Inst. Tech. **20**, 337–354 (1941). [MF 6105]

The differential analyzer is a machine for obtaining numerical solutions to ordinary differential equations, and consists fundamentally of four types of mechanical elements, integrators, input tables, adders and ratio or gear boxes, whose various input and output shafts can be interconnected in such a way that their angular rotations are constrained to generate continuously the values of the variables and functions in the equation to be solved. "The most important results [mostly given in the form of theorems with proofs] deal with conditions under which functions of one or more variables can be generated, and conditions under which ordinary differential equations can be solved. Some attention is given to approximation of functions (which cannot be generated exactly), approximation of gear ratios and automatic speed control."

H. L. Hazen (Cambridge, Mass.).

McPherson, J. C. Mathematical operations with punched cards. J. Amer. Statist. Assoc. **37**, 275–281 (1942). [MF 6802]

Hartkemeier, Harry Pelle and Miller, Herman E. Obtaining differences from punched cards. J. Amer. Statist. Assoc. **37**, 285–287 (1942). [MF 6803]

Lilley, S. Mathematical machines. Nature **149**, 462–465 (1942). [MF 6679]

***Comrie, L. J. The Twin Marchant Calculating Machine and Its Application to Survey Problems.** Scientific Computing Service Limited, London, 1942. 40 pp.+3 plates. 10s.

Some of these notes have already appeared in "On the Application of the Brunsviga Twin 13Z Calculating Machine to Survey Problems" [first edition, 1938; second edition, 1940]. Others are taken from paragraphs contributed by the author to "Machine Calculation" [Scientific Computing Service, Ltd., London, 1939].

Kneissl, M. Näherungsverfahren zum Zusammenschluss von Dreiecksnetzen. Allg. Vermessgs.-Nachr. **53**, 143–158, 177–183 (1941). [MF 6857]

These are the second and last parts of the paper, the first and third parts of which appeared in the same journal **53**, 121–131, 169–174 (1941) [these Rev. **2**, 367].

Spencer, Domina Eberle. Calculation of illumination from triangular sources. J. Opt. Soc. Amer. **32**, 274–281 (1942). [MF 6609]

A rectilinear source of light with constant helios H can be broken up into triangular sources, each of which produce a certain flux density D at a point P of a given surface such as a drawing board. The planes determined by P and the sides of the triangle cut out a spherical triangle ABC on the unit sphere about P, in terms of whose sides and angles the flux density may be computed. The computation is performed by nomograms, including a haversine scale to solve the spherical triangle, and various sine and cosine scales set up as alignment charts. Numerical cases are also considered.

J. S. Frame (Providence, R. I.).

de Sampaio Pacheco, Murillo. On a class of abaci. Anais Acad. Brasil. Sci. **12**, 331–337 (1940). (Portuguese) [MF 6579]

The conditions are obtained for a nomographic chart with six variables on six general curved scales and three index lines all passing through a common unspecified variable point.

P. W. Ketchum (Urbana, Ill.).

MECHANICS

Dynamics

Weber, Moritz. Die Lagrangeschen Bewegungsgleichungen für allgemeine Koordinaten. Z. Verein. Deutsch. Ingenieure **85**, 471–480 (1941). [MF 6960]

Expository article intended for engineers.

Birkhoff, George D. Some unsolved problems of theoretical dynamics. Science **94**, 598–600 (1941). [MF 5973]

This paper is itself an abstract of a paper read at a fiftieth anniversary symposium of the University of Chicago. All that can be said here is that it presents a list of some seventeen unsolved problems of dynamics. The first ten of these are formulated in terms of abstract spaces. The eleventh is concerned with the extension of certain results of Sundman on the three body problem to the motion of a gas. The last six are concerned with n-dimensional spaces and are of a topological nature. The last two of these are concerned with the highly interesting conservative transformation of a two-dimensional ring into itself in cases where Poincaré's "last geometric theorem" does not apply.

D. C. Lewis (Durham, N. H.).

von Krbek, F. Die Integralprinzipe der Mechanik. Acta Math. **74**, 101–108 (1941). [MF 6513]

The well-known variational equation for a general dynamical system is put into the form

$$\int_{t_1}^{t_2}\left(\Delta T+2T\frac{d}{dt}\Delta t+\dot{T}\Delta t+\delta A\right)dt=\left[2T\Delta t+\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2}.$$

The correspondence between points on the natural and varied trajectories is general. The variation in passing to a corresponding point is indicated by Δ; δ indicates the variation to a simultaneous configuration, and δA is work done.

J. L. Synge (Toronto, Ont.).

Sonnino, Sergio. Sulla integrazione dell'equazione di Hamilton-Jacobi per separazione di variabili. Anais Acad. Brasil. Sci. **12**, 45–50 (1940). [MF 6570]

The main theorem of this paper is the following: A necessary condition for the integration of the Hamilton-Jacobi partial differential equation by separation of variables, assuming that the kinetic energy is of the form $\frac{1}{2}B(q_1, \cdots, q_n)\sum_{i=1}^{n}A_i(q_i)\dot{q}_i^2$, is that there exist functions $B_i(q_i)$ and $U_i(q_i)$ such that $B(q_1, \cdots, q_n)=\sum_{i=1}^{n}B_i(q_i)$ and such that

the potential energy is given by $\sum_{i=1}^{n} U_i(q_i)/\sum_{i=1}^{n} B_i(q_i)$. That this condition is also sufficient is well known and was proved by Liouville. The author's proof of the necessity depends in a very direct way on some very general conditions given by Levi-Civita in a well-known letter to Stäckel [Math. Ann. 59, 383–397 (1904)]. *D. C. Lewis.*

Sonnino, Sergio. Sulla integrazione di una equazione di Levi Civita che interessa il problema della separazione di variabili dell'equazione di Hamilton-Jacobi. Anais Acad. Brasil. Sci. 12, 269–275 (1940). [MF 6576]

One of Levi-Civita's conditions referred to in the paper reviewed above is a system of partial differential equations to be satisfied by the kinetic energy K, which is assumed to be a positive definite quadratic form in the momenta $p_1, \cdots, p_n$. The present paper discusses solutions of this system having the form $K=\varphi(c_1p_1+q_1, \cdots, c_np_n+q_n)$, where q_i is the coordinate associated with the momentum p_i and where c_i may be a function of all the q's except q_i. There are numerous misprints. *D. C. Lewis* (Durham, N. H.).

Madhava Rao, B. S. On the reduction of dynamical equations to the Lagrangian form. Proc. Benares Math. Soc. (N.S.) 2, 53–59 (1940). [MF 6648]

The author discusses conditions under which a system of equations of the form

$$\frac{d^2q_k}{dt^2}=f_k\left(\frac{dq_1}{dt}, \cdots, \frac{dq_n}{dt}, q_1, \cdots, q_n, t\right), \quad k=1, 2, \cdots, n,$$

is equivalent to a Lagrangian system. He uses various hypotheses concerning the nature of the Lagrangian function as well as that of the f's. The first part of the paper is purely expository, leading by slightly modified methods to known results. In the last part of the paper it is shown that under certain conditions the reduction to Lagrangian form depends on the problem of finding last multipliers. *D. C. Lewis* (Durham, N. H.).

van den Dungen, F.-H. Sur les équations aux dérivées partielles du second ordre associées aux mouvements de la mécanique classique. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 279–287 (1941). [MF 6889]

The second order partial differential equations referred to in the title (with independent variables $q_1, \cdots, q_n, t, S$ and dependent variable u) are those equations whose characteristic surfaces $S=S(q_1, \cdots, q_n, t)$ are obtainable by solving the usual Hamilton-Jacobi first order partial differential equation

$$\frac{\partial S}{\partial t}+H(q_1, \cdots, q_n, \partial S/\partial q_1, \cdots, \partial S/\partial q_n, t)=0.$$

One form of such a second order equation is closely related to Schrödinger's wave equation. The purpose of the paper is to show a certain duality between wave mechanics and classical mechanics which can be attained by introducing a new independent variable, namely, the quantity S. *D. C. Lewis* (Durham, N. H.).

Pipes, Louis A. Operational analysis of non-linear dynamical systems. J. Appl. Phys. 13, 117–128 (1942).

The purpose of the present note is to show that the method of Laplace transforms can be applied in solving various nonlinear problems of the theory of oscillators with one degree of freedom and nonlinear electric circuits. The method used by the author consists in developing the solution, or rather its Laplace transform, in power series of a small parameter and equating the coefficients up to the third power of the parameter. Although no discussion of convergence or estimates of error are given, it is shown that approximations so obtained are useful and could be evaluated comparatively easily. *J. D. Tamarkin.*

Tesch, H. Der Einfluss der Elastizität der Schneide und Unterlage eines Pendels auf die Schwingungszeit. Z. Geophys. 16, 289–309 (1940). [MF 6783]

Astronomy

Williams, K. P. and Clemence, G. M. Note on Newcomb's tables of the three inner planets. Astr. J. 50, 15–17 (1942). [MF 6796]

***Møller, Jens P. On the solution of Kepler's equation.** Astronomical Papers dedicated to Elis Strömgren, pp. 163–174. Einar Munksgaard, Copenhagen, 1940.

For values of e near 1 the usual methods of solving Kepler's equation, using a calculating machine, are troublesome and inaccurate. After discussing various methods, the author decides to write the equation in the form: $M^0=(1-e)^0 \sin E=S^0(E)$ and constructs a table giving the value of $S^0(E)$ as a function of E. The value of $\cos E$ is also given in the table, so that the coordinates in the plane of the orbit may be found. He gives an example showing how his table is used. *H. E. Buchanan* (New Orleans, La.).

Rabe, E. Periodische Lösungen für die Bewegung eines Doppelplanetoiden. Astr. Nachr. 271, 181–185 (1941). [MF 6797]

The author develops a variation orbit for the motion in an asteroid system consisting of two components of comparable mass. The reduction to the restricted problem is accomplished by a modification of the equations of motion similar to that introduced in the lunar theory; the results are then obtained by direct substitution into the literal developments given by G. W. Hill [Collected Mathematical Works, Carnegie Institution of Washington, Washington, D. C., 1905, v. 1, pp. 284–335]. *D. Brouwer.*

Kaplan, Wilfred. Topology of the two-body problem. Amer. Math. Monthly 49, 316–323 (1942). [MF 6695]

It is shown in detail that the manifold E consisting of points of phase space corresponding to a fixed negative value of the energy constant is topologically equivalent to Euclidean 3-space minus a line. If in addition the area constant is fixed a manifold AE is obtained which is in general equivalent to a torus. It is also shown how the individual trajectories fill AE and E, and how, by varying the area constant, the tori AE fill the complete space E. *D. C. Lewis* (Durham, N. H.).

Batyrev, A. A. On the shape of trajectories in problems of two and three bodies with masses changing according to the Meshchersky law. Astr. J. Soviet Union [Astr. Zhurnal] 18, 343–346 (1941). (Russian. English summary) [MF 6640]

Through a suitable change of the coordinates and of the time parameter the problem of two bodies with masses variable according to the law $M(t)=(a+bt)^{-1}$ is reduced to the classical case with constant mass. The equations of

orbits corresponding to conic sections in the classical problem are given. *I. Opatowski* (Chicago, Ill.).

Pylarinos, O. Über die Lagrangeschen Fälle im verallgemeinerten Dreikörperproblem. Math. Z. 47, 357–372 (1941). [MF 6723]

The author studies the particular solutions of the three body problem in which the ratios of the mutual distances remain constant in the case of a generalized law of attraction. Initially the only assumption is that the forces act along the lines joining the bodies. It is shown that the solutions in which the three bodies lie in a straight line are necessarily planar, that is, the motions take place in a plane that is fixed in an inertial coordinate system whose origin coincides with the center of mass. This is not generally the case with the triangular solutions. Let the three forces acting upon the three material points P_1, P_2, P_3 intersect in a point D. Its position vector can be expressed linearly in terms of the position vectors of P_1, P_2, P_3 with coefficients $\kappa_1, \kappa_2, \kappa_3$. The necessary and sufficient condition for a planar solution is that D coincides with the center of mass. In that case $\kappa_1, \kappa_2, \kappa_3$ are independent of the time.

Further theorems are derived, the principal of which is that if the triangle formed by P_1, P_2, P_3 remains similar throughout the motion, and if $\kappa_1, \kappa_2, \kappa_3$ are independent of the time, then the plane of P_1, P_2, P_3 is either fixed or rotates about an axis fixed in space, and that coincides with one of the two principal inertial axes of the system that lie in the plane of P_1, P_2, P_3. Cases of forces proportional to different powers of the mutual distances are considered. It is shown that here generally only the planar solution exists, but that in the case of attraction inversely proportional to the third power of the distance there exists in addition a nonplanar solution. [The paper was written before the appearance of Wintner's, The Analytical Foundations of Celestial Mechanics, Princeton University Press, Princeton, N. J., 1941, in particular §§369–374; these Rev. 3, 215.] *D. Brouwer* (New Haven, Conn.).

Schwarzschild, Martin. On stellar rotation. Astrophys. J. 95, 441–453 (1942). [MF 6684]

The problem treated is that of the equilibrium of a star with nonvanishing angular momentum. The rotation is not assumed to be uniform. The only effective viscosity is supposed to be that arising from turbulent motion in the central core, and in the shallow hydrogen-convection layer at the surface. The viscosity has been taken into account by assuming the core to rotate solidly, and by demanding the velocity gradient at the surface to satisfy the usual condition at a free surface of a viscous fluid. Apart from these conditions the equations are those of a nonviscous fluid. The first order equations are developed for a slowly rotating star of the standard model. It is found that the viscosity-surface condition leads necessarily to a solid body rotation in the first order approximation. The perturbation equations are integrated for the uniformly rotating standard model.

Appendix A gives a justification for neglecting second power of the perturbations even at the surface, where the quantities themselves vanish. Appendix B shows that higher harmonics than the second must vanish in the expression for the perturbed potential. Appendix C removes the apparent contradiction between von Zeipel's theorem and the result that the standard model must rotate (to the first order) rigidly. Appendix D shows the uniqueness of the solution obtained. The correctness of the application of the viscosity-surface condition, which is decisive for the development, is doubtful, and must be investigated. *G. Randers* (Chicago, Ill.).

Ganguly, H. K. On the equilibrium configuration of a rotating fluid envelope surrounding a rotating core. Bull. Calcutta Math. Soc. 31, 127–136 (1939). [MF 6147]

This paper studies the distribution of angular velocities in a body consisting of a spheroidal fluid core covered by another spheroidal fluid envelope of different density, both rotating about a common axis. If ω_i and ω_e be the angular velocity of the inner and outer fluids, respectively, at a point of the common surface, it is proved that $\omega_i^2-\omega_e^2$ is monotone increasing from the poles to the equator. If the outer boundary is kept fixed, while the core is changed from an extremely flat spheroid to a more nearly spherical shape, a critical stage will be reached such that in every configuration from the extremely flat to the critical one $\omega_i>\omega_e$. For a series of configurations after this critical one, $\omega_i<\omega_e$ in a region near the poles while around the equator it may be that $\omega_i>\omega_e$. However, after the core becomes sufficiently close to a sphere, then for all points on the boundary of the common surface $\omega_i<\omega_e$. *B. Friedman* (Chicago, Ill.).

Haalck, H. Das Gleichgewicht der Kräfte im Innern des Erdkerns und die sich daraus ergebenden Folgerungen. Z. Geophys. 17, 135–146 (1941). [MF 6668]

Mustel, E. R. Radiative equilibrium of the atmospheres of hot stars. Astr. J. Soviet Union [Astr. Zhurnal] 17, 12–40 (1940). (English. Russian summary) [MF 6677]

Mustel, E. R. Distribution of energy in the continuous spectrum of stars of early classes. Astr. J. Soviet Union [Astr. Zhurnal] 18, 297–311 (1941). (English. Russian summary) [MF 6639]

Chandrasekhar, S. A statistical theory of stellar encounters. Astrophys. J. 94, 511–524 (1941). [MF 6911]

Assume a uniform Poisson distribution of stars in space (the limiting distribution of NR^3 stars of equal mass, uniformly and independently distributed over a sphere of radius R, when $R\to\infty$). This distribution of stars determines a statistical distribution of gravitational force F on a star at a given point O. Holtsmark [Ann. Physik (4) 58, 577–630 (1919)] investigated an analogous problem, finding the distribution of $|F|$. (The distribution of F is a stable distribution with exponent 3/2.) Chandrasekhar shows that the Holtsmark distribution is very nearly the distribution of force due only to the star nearest O. He then modifies the distributions to obtain lower probabilities for high field strengths by imposing conditions restricting the possible closeness of the stars to O. As a first approximation to the actual stellar motion caused by the random gravitational field, the dispersion of velocity change of a star in a time interval t, $\overline{\Delta v^2}$ is calculated, assuming successive uniform accelerations (determined by F) over short time intervals, in random directions. This evaluation is applied to obtain the time of relaxation of a stellar system (the time required for $\overline{\Delta v^2}$ to become of the same order as $\overline{v^2}$). *J. L. Doob.*

Chandrasekhar, S. and von Neumann, J. The statistics of the gravitational field arising from a random distribution of stars. I. The speed of fluctuations. Astrophys. J. 95, 489–531 (1942). [MF 6685]

The authors investigate the joint probability distribution of the vectors F, dF/dt, where F is the force acting on a

given star. This force is considered as due to stars distributed as described in the preceding review, each with a spherical distribution of velocities about the origin in velocity space, independent of the position of the star. The method used is that of characteristic functions. The distribution of F alone has already been discussed by Chandrasekhar [cf. the preceding review]. In the present paper, which is a step towards derivation of the transition probabilities of F as a function of time, an expression for the joint density function of F, dF/dt is found. Using this, the second moment of dF/dt for given F, that of F for given dF/dt, and that of dF/dt are found. Using these moments, it is possible to discuss the probable characteristics of field strength transitions, questions of mean life of a given state F, etc. *J. L. Doob* (Washington, D. C.).

Hydrodynamics, aerodynamics

Lin, C. C. On the motion of vortices in two dimensions. I. Existence of the Kirchhoff-Routh function. Proc. Nat. Acad. Sci. U. S. A. **27**, 570–575 (1941). [MF 5906]

Suppose there are n vortices of strengths k_i at (x_i, y_i), $i=1, \cdots, n$, in a region made multiply connected by the presence of barriers. Koebe has demonstrated the existence of a Green's function satisfying $\int_\Gamma (\partial G/\partial n)ds=0$, with the usual notation, where Γ is any one of a set of base cycles for the region. [The author's comment in footnote 9 is not correct, but this does not affect the results of the paper.] If the region extends to infinity the usual order conditions are imposed on G. The stream function is the sum of a component, namely, $\sum_1^n k_i G(x, y; x_i, y_i)$, depending on the vortices and a harmonic function $\psi_0(x, y)$ independent of the vortices and determined by the circulation about the barriers and a condition at infinity. The Hamiltonian governing the notion of the vortices is termed the Kirchhoff-Routh function and is given explicitly by

$$W=\sum k_i\psi_0(x_i, y_i)+\sum k_i k_j G(x_i, y_i; x_j, y_j) + \sum k_i^2 L_{x,y\to x_i,y_i}(G(x, y; x_i, y_i)-\log r_i).$$

Thus

$$\frac{dk_i^{\frac{1}{2}}x_i}{dt}=-\frac{\partial W}{\partial k_i^{\frac{1}{2}}y_i}, \quad \frac{dk_i^{\frac{1}{2}}y_i}{dt}=\frac{\partial W}{\partial k_i^{\frac{1}{2}}x_i}.$$

D. G. Bourgin (Urbana, Ill.).

Lin, C. C. On the motion of vortices in two dimensions. II. Some further investigations on the Kirchhoff-Routh function. Proc. Nat. Acad. Sci. U. S. A. **27**, 575–577 (1941). [MF 5907]

Under a conformal transformation $z'=z'(z)$ the Kirchhoff-Routh function defined in the preceding review becomes

$$W'=W+\sum \frac{k_i^2}{4\pi}\log\left|\frac{dz}{dz'}\right|_{x,y=x_i,y_i}.$$

D. G. Bourgin (Urbana, Ill.).

Bechert, Karl. Über die Ausbreitung von Zylinder- und Kugelwellen in reibungsfreien Gasen und Flüssigkeiten. Ann. Physik (5) **39**, 169–202 (1941). [MF 6847]

If u is the radial velocity, which is directed along the radius from the origin in the case of spherical waves ($k=3$) and at right angles to the axis of symmetry in the case of cylindrical waves ($k=2$), the equations of motion

$$u_t+uu_r+\rho^{-1}p_r=0, \quad \rho_t+u\rho_r+\rho u_r+(k-1)\rho u/r=0$$

are transformed by means of the substitution

$$\xi=2a^{\frac{1}{2}n-\frac{1}{2}}[k(n-1)]^{-\frac{1}{2}}, \quad \eta=u(n-1)^{\frac{1}{2}},$$

where a and n are the constants occurring in the polytropic law $p-p_0=a^2\rho^n/n$, $n\neq 1$, by which the pressure p is supposed to be expressed as a function of ρ only. If $n=1+m^2$ the new equations are

$$m\eta_t+\eta\eta_r+\tfrac{1}{2}m^2k\xi\xi_r=0, \quad m\xi_t+\eta\xi_r+\tfrac{1}{2}m^2\xi\eta_r+\tfrac{1}{2}(k-1)m^2\xi\eta/r=0.$$

It is assumed in Part II that the density ρ depends only on the velocity u, that is, that $\xi=F(\eta)$. The problem then reduces to the solution of the ordinary differential equation

$$m^2(k-1)\phi(q-\phi)dq/d\phi+q^2(km^2-n-1) +m^2\phi q-m^2k\phi q^2+2=0,$$

where ϕ, q depend on ξ, η, r and t by means of the equations

$$\eta=C\exp\left(\int^\phi \frac{d\phi}{(q-\phi)}\right); \quad \xi=\phi\eta; \quad \frac{mr}{t+\tau}=\eta(1+\tfrac{1}{2}km^2q\phi).$$

Prescribed values of ρ and u progress in this type of wave with a speed not depending on the time; in adiabatic processes in gases this speed is $dr/dt=u+ck^{\frac{1}{2}}d\xi/d\eta$, where $c=ar^{\frac{1}{2}m^2}$ is the velocity of sound. Shock waves occur with the connection $\xi=F(\eta)$ either for all values of t or not at all. A further solution $m/\eta=1+\frac{1}{2}km^2(t+\tau)/r$, $\xi=\pm\eta$, is furnished by the singular solution of the differential equation. When $n=-1$ the differential equation for q has a solution in a closed form

$$(kq^2-1)^{k-1}\phi^{2k}=C(kq-1)^{2k-2},$$

where C is a constant. In other cases it is reduced to a form in which a graphical integration is possible by the method of isoclines. This form is

$$k(w-1)(zv-w^2)dv/dw=w(k-1)(2w+km^2)v.$$

In Part III the more general assumption $\xi=r^{-\alpha}F(r^\alpha\eta)$ is found to yield solutions for arbitrary values of α. The general solution of the resulting equation gives $r^\alpha\eta$, $r^\alpha\xi$ and $r^{\alpha+1}/(t+\tau)$ as functions of a parameter ϕ and 4 arbitrary constants, provided $\alpha\neq -1$. A differential equation of the first order for a function $q(\phi)$ must again be solved. Prescribed values of $r^\alpha\xi$ and $r^\alpha\eta$ are now propagated with a speed depending on r like $r^{-\alpha}$ and on t like $(t+\tau)^{-\alpha/(\alpha+1)}$, where τ is a constant. These waves generally suffer distortion as they travel. Shock waves occur either for all values of t or not at all. When $\alpha\neq 0$ there are no singular integrals. In the exceptional case $\alpha=-1$ prescribed values of η/r and ξ/r are propagated at a speed proportional to r and depending exponentially on the time t. The waves are again distorted as they travel and again shock waves occur either for all values of t or not at all. The paper ends with some special and general remarks on the differential equations.

H. Bateman (Pasadena, Calif.).

Bechert, Karl. Über die Differentialgleichungen der Wellenausbreitung in Gasen. Ann. Physik (5) **39**, 357–372 (1941). [MF 6849]

After some general remarks on the type of inverse calculation which is needed to solve the initial value problem for the nonlinear partial differential equations for the propagation of waves of finite amplitude in gases, the features of oscillations of finite amplitude of a gas between fixed walls are mentioned. A similarity transformation is used for the equations for plane, cylindrical and spherical waves in order to obtain special solutions like that found in the previous paper. This method is capable of generalization. Use is then

made of a substitution $m=\int_{x_0}^{x}\rho dx$ recommended by H. Marx, the quantities m, t being used in place of x and t in the treatment of plane waves. This plan can also be adopted when the polytropic law for the relation between pressure p and the density ρ is not adopted. *H. Bateman.*

Ballabh, Ram. Superposable fluid motions. Proc. Benares Math. Soc. (N.S.) **2**, 69–79 (1940). [MF 6650]

This paper is concerned with incompressible viscous fluids, for which the differential equations of motion are well known to be nonlinear. Two different definitions of superposability are given, which, however, are shown to lead to the same theory if the external forces are conservative. General necessary and sufficient conditions are given that two solutions be superposable or that one solution be self-superposable. It is always possible to express these conditions independently of the kinematic coefficient of viscosity ν, as one might well expect from the fact that the terms in the equations of motion which involve ν are linear. It thus turns out that our conditions for superposability (particularly self-superposability) yield generalizations for viscous fluids of certain well-known theorems for nonviscous fluids. On the other hand, one of the neatest forms of the conditions does involve ν and is to the effect that a motion is self-superposable if, and only if, the components of vorticity are solutions of the heat equation $\partial\theta/\partial t=\nu\nabla^2\theta$.

Two particular sufficient conditions for superposability are given, one of which is to the effect that the stream lines of each motion are the same as its vortex lines. This latter condition is discussed at some length and the author attempts to show that if the coefficient of proportionality λ between the components of vorticity and the components of velocity is independent of x, y, z it is also independent of the time. The argument here does not appear to be correct, as the author has not allowed for sufficient generality in the solution of the above mentioned heat equation.

D. C. Lewis (Durham, N. H.).

Ballabh, Ram. Self superposable motions of the type $\xi=\lambda u$ etc. Proc. Benares Math. Soc. (N.S.) **2**, 85–89 (1940). [MF 6652]

A continuation of the preceding paper in which it is shown that, if λ is independent of x and y, it must also be independent of z. *D. C. Lewis* (Durham, N. H.).

Hantzsche, W. und Wendt, H. Die kompressible Potentialströmung um eine Schar von nichtangestellten symmetrischen Zylindern im Kanal. Luftfahrtforschung **18**, 311–316 (1941). [MF 6701]

The Janzen-Rayleigh method of successive approximations is employed to calculate the velocity potential $\Phi(x, y)$ of an irrotational compressible flow: $\Phi=\Phi_0+\Phi_1M^2+\cdots$, where M is the Mach number of the undisturbed flow at infinity. Following Imai and Aihara [Reports Aeronaut. Research Inst. Tokyo Imp. Univ. **15**, 185–212 (1940)], the authors obtain Φ_1 in the form

$$\Phi_1=\frac{1}{4U^2}\Re\left\{\overline{\frac{df(z)}{dz}}\int\left(\frac{df}{dz}\right)^2dz+F(z)+G(\bar{z})\right\},$$

where $z=x+iy$, $\bar{z}=x-iy$, $\Phi_0=\Re\{f(z)\}$ is the velocity potential of the corresponding incompressible flow, U is the velocity at infinity and $F(z)$ and $G(\bar{z})$ are arbitrary functions.

For the case of a symmetrical cylinder located symmetrically in the middle of a straight wind channel the transformation $\zeta=e^z$ is employed. The authors consider the family of cylinders that are transformed into circles in the ζ plane. These are nearly-elliptical cylinders, characterized by the fact that in the incompressible flow they all have the maximum velocity $2U$ at the points of their contours nearest the walls (U is the channel speed at infinity). For such cylinders the function Φ_1 is obtained in closed form. Numerical calculations including the terms in M^2 have been carried out by the authors for this family of cylinders, and the compressibility effect on the maximum velocities at the cylinder and at the walls is shown. The critical value of M at which the velocity of sound first appears on the cylinder is also given. For one cylinder of the family the velocity distribution along the contour and the velocity profile in the restricted section of the channel are shown graphically for three values of M. *W. R. Sears* (Inglewood, Calif.).

Lamla, Ernst. On the symmetrical potential flow of compressible fluid past a circular cylinder in the tunnel in the subcritical zone. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1018**, 11 pp. (1942). [MF 6795]

Translation of the author's paper in Luftfahrtforschung **17**, 329–331 (1940); cf. these Rev. **2**, 266.

Keune, F. Two-dimensional potential flow past an ordinary thick wing profile. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1023**, 52 pp. (1942). (32 figures) [MF 7019]

Translation of a paper in Jahrbuch 1938 der Deutschen Luftfahrtforschung, pp. I 3–I 26.

Tollmien, W. Grenzlinien adiabatischer Potentialströmungen. Z. Angew. Math. Mech. **21**, 140–152 (1941).

The author [Z. Angew. Math. Mech. **17**, 117–136 (1937)] and F. Ringleb [Z. Angew. Math. Mech. **20**, 185–198 (1940); Deutsche Math. **5**, 377–384 (1941); cf. these Rev. **2**, 169, 267] have discovered certain lines of singularities or "limiting lines" beyond which the adiabatic irrotational flow of a compressible fluid apparently cannot progress. To investigate this phenomenon the author employs the Molenbroek-Tschapligin transformation from the x, y (physical) plane to the q, ϑ (hodograph) plane, where $q=(\psi_x+\psi_y)^{\frac{1}{2}}$, $\vartheta=\tan^{-1}(-\psi_x/\psi_y)$, and ψ is the stream function. Considering q and ϑ as independent variables the differential equation for ψ becomes linear. Ringleb also used this transformation and found that a streamline ($\psi=$constant), when transformed from the hodograph plane to the physical plane, has a cusp at the "limiting line," reverses its direction, and crosses the flow in a physically impossible manner. He also discovered that the acceleration is infinite at the cusp.

The author discusses the Molenbroek-Tschapligin transformation in detail, and also points out a class of flows (wherein $q=q(\vartheta)$) that are "lost" in this transformation. These occur when the Jacobian $\partial(q, \vartheta)/\partial(x, y)$ vanishes in a region of nonvanishing area in the x, y plane. The "limiting lines" are lines along which this Jacobian vanishes, that is,

$$\psi_q^2-(1/a^2-1/q^2)\psi_\vartheta^2=0,$$

where a is the local velocity of sound. This can occur only in a region of supersonic flow; in fact, in the hodograph plane such a line connects points where the streamlines are tangent to the characteristics of the differential equation. The author obtains Ringleb's results and shows that in the physical plane the line is an envelope of "Mach wavelets" (the transformed characteristics). In the closing paragraphs

he deduces that the irrotational adiabatic flow not only cannot progress beyond a limiting line but must actually break down before reaching the line.

[Reviewer's note: The results presented in this paper are substantially parallel to those of von Kármán published concurrently [J. Aeronaut. Sci. 8, 337–356 (1941); these Rev. 3, 220].] *W. R. Sears* (Inglewood, Calif.).

Pretsch, J. Über die Stabilität der Laminarströmung um eine Kugel. Luftfahrtforschung **18**, 341–344 (1941). [MF 6428]

The stability of two dimensional flow in the laminar boundary layer by the method of small oscillations has been investigated by Tollmien, Schlichting and others. The author extends these investigations to the flow past a solid sphere. The velocity field is assumed to consist of a mean motion symmetrical about one axis with the tangential component U and the normal component V. Superimposed are the small components of the perturbation velocity u^* and v^* which are derived from the stream function

$$\psi(s, n, t)=\varphi(s, n, t)e^{i(2\pi/\lambda)(s-ct)},$$

where λ is the wavelength, c the phase velocity of the perturbation, s and n denote the distance along the circumference measured from the stagnation point and the distance normal from the surface, respectively. In order to show that the particular solutions of this problem are the same as Tollmien's solutions for plane motion the author proves that (1) the perturbation equation in the region where the friction can be neglected is reduced to

$$(U-c)[\varphi''-(2\pi/\lambda)^2\varphi]-U''\varphi=0,$$

which is the well-known "frictionless" equation of the plane case; (2) the behavior of the equation in the so-called critical layer, that is, where the phase velocity c is equal to the mean velocity U, is the same as in Tollmien's investigation; (3) very close to the wall where the influence of friction becomes predominant the equations in both cases are also identical. Since the boundary conditions are the same it is thus proved that the procedure which was developed for solving stability problems in plane motion can be applied to the flow past a sphere. The author believes that this result might hold for all problems of axial symmetry.

As an application the author computes the stability limit of the velocity profile at the stagnation point of a sphere which has been given by Homann. The results are presented in a diagram of δ^*/λ versus R_{δ^*}, where R_{δ^*} is the Reynolds number based on the displacement thickness of the boundary layer δ^*. For comparison an investigation of the plane stagnation point profile as computed by Hiementz is also shown. The critical Reynolds number was found to be 4200 for the flow with axial symmetry and 12100 for the plane motion. Thus the latter appears to be much more stable.

H. W. Liepmann (Pasadena, Calif.).

Pretsch, J. The stability of laminar flow past a sphere. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1017**, 7 pp. (1942). (1 plate) [MF 6794]

Translation of the paper reviewed above.

Emmons, H. W. and Brainerd, J. G. Temperature effects in a laminar compressible-fluid boundary layer along a flat plate. J. Appl. Mech. **8**, A-105–A-110 (1941). [MF 6231]

Solutions are obtained, by means of the differential analyzer, for the two-dimensional boundary layer problem of the steady flow of a perfect gas along a flat, insulated boundary. Typical velocity and temperature distributions and the drag on the boundary are given, as functions of the parameters involved, in the form of diagrams. The mathematical problem is the integration, subject to appropriate boundary conditions, of the system of equations

$$T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y},$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu T}{\rho_1 T_1}\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\gamma\rho}{\gamma-1}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\mu\left(\frac{\partial u}{\partial y}\right)^2+k\frac{\partial^2 T}{\partial y^2},$$

where u and v are velocity components, T the temperature, μ the viscosity, ρ the density, k the thermal conductivity, γ the ratio of the specific heats c_p/c_v, and the subscripts designate the values of ρ and T at infinity. Introducing, as was first done by Blasius, a variable η proportional to $y/\sqrt{x}$, the system of partial differential equations is reduced to a system of ordinary differential equations by the assumption that all variables depend on η only, and this system is integrated mechanically.

The authors assume $\mu=$const., while in earlier work by A. Busemann [Z. Angew. Math. Mech. **15**, 23–25 (1935)] and von Kármán and Tsien [J. Aeronaut. Sci. **5**, 227–232 (1938)] there was assumed $\mu=$const. $T^{\frac{1}{2}}$ and $\mu=$const. $T^{0.76}$, respectively. The authors' solutions are obtained for various values of the parameter $\mu c_p/k$, while in the earlier work referred to the calculations are restricted to the case $\mu c_p/k=1$.

E. Reissner (Cambridge, Mass.).

Weyl, Hermann. On the differential equations of the simplest boundary-layer problems. Ann. of Math. (2) **43**, 381–407 (1942). [MF 6468]

The nonlinear boundary-value problem

$$(A_\lambda)\qquad \begin{cases} w'''+2ww''+2\lambda(k^2-w'^2)=0 & \text{for } z\geqq 0,\\ w(0)=w'(0)=0,\quad w'(z)\to k & \text{for } z\to\infty, \end{cases}$$

where $w(z)$ is a function of a real variable $z\geqq 0$, and $k>0$ and $\lambda\geqq 0$ are constants, arises in the theory of viscous fluids. It has two interpretations: (1) for $\lambda=\frac{1}{2}$ it describes approximately the flow in the boundary layer of a blunt-nosed body of cylindrical symmetry, near the stagnation point, and (2) it describes the boundary-layer flow along a two-dimensional prow of angle $\pi\lambda$.

After setting up the problem (A_λ) in both of these applications, the author obtains solutions in the cases $\lambda=0$ (Blasius' problem), $\lambda=\frac{1}{2}$, $k=0$ (Goldstein's wake problem) and $\lambda=\frac{1}{2}$, $k>0$ (Homann's problem) by a method applied in an earlier note [Proc. Nat. Acad. Sci. U. S. A. **27**, 578–583 (1941); these Rev. **3**, 220]. The method is one of successive approximations and converges rapidly. Convergence and uniqueness are proved. For the case of arbitrary $\lambda\geqq 0$, the method of fixed points of functional transformations is employed, as in another earlier note [Proc. Nat. Acad. Sci. U. S. A. **28**, 100–102 (1942); these Rev. **3**, 221]. The functional operator Φ_λ that carries a function $g=f''$ into φ is defined by

$$f(z)=\int_0^z (z-\zeta)g(\zeta)d\zeta,$$

$$\begin{cases} \varphi''+2f\varphi'+2(1-2\lambda)f'\varphi=0,\\ \varphi(0)=1,\quad \varphi(\infty)=0. \end{cases}$$

The solution of (A_λ) is given if a fixed element of this transformation can be found: $g=\Phi_\lambda\{g\}$. A series of lemmas is used to prove the existence of a solution. The general theory is attributed to Birkhoff and Kellogg [Trans. Amer. Math. Soc. **23**, 96–115 (1922)] and Schauder and Leray [Studia Math. **2**, 171–180 (1930); Ann. École Norm. **51**, 45–78 (1934)]. *W. R. Sears* (Inglewood, Calif.).

Christopherson, Derman G. A new mathematical method for the solution of film lubrication problems. Inst. Mech. Engrs. J. Proc. **146**, 126–135 (1941). [MF 6245]

The paper applies the "method of relaxation" as suggested by Southwell and adapted for problems of the present type by Christopherson and Southwell [Proc. Roy. Soc. London. Ser. A. **168**, 317–350 (1938)] to the solution of Osborne Reynolds' equation of the lubrication problem. The great adaptability of the method is shown. Not only such difficulties as arise from the occurrence of a divergent film in journal bearings can be mastered satisfactorily, but even the variation of viscosity with pressure and temperature can be taken into account. A comparison shows that the usual treatment, which assumes viscosity to be constant, may lead to good results as to magnitude of load, quantity of flow through the bearing and temperature rise, but is apt to give quite unsatisfactory results concerning the frictional resistance and the position of the load.

P. Nemenyi (Fort Collins, Colo.).

Mohr, Ernst. Über den Navier-Stokesschen Spannungsansatz für zähe Flüssigkeitsströmungen. Luftfahrtforschung **18**, 327–330 (1941). [MF 6703]

The author discusses the molecular theory of viscosity, in terms of momentum transport, and suggests an unsymmetric stress tensor to replace the usual symmetric one.

J. L. Doob (Washington, D. C.).

Chou, P. Y. On an extension of Reynolds' method of finding apparent stress and the nature of turbulence. Chinese J. Phys. **4**, 1–33 (1940). [MF 6177]

The paper starts with a review of the conventional approach to turbulence in tensor notation; time-periodicity of the velocity fluctuation w_i and the pressure fluctuation p' is assumed. It is possible to operate on the Navier-Stokes equations in the following alternative ways: (i) take the time-average, (ii) multiply by w_j and then average, (iii) multiply by w_jw_k and then average, and so on. (i) gives the Reynolds equations, (ii) gives equations involving correlations up to the third order and (iii) gives equations involving correlations up to the fourth order. Each of these plans yields an indeterminate problem, since there are more correlations than equations. The author follows plan (iii), and gets rid of the indeterminacy by the following auxiliary hypotheses:

(a) $\overline{w_iw_jw_kw_l}=\frac{1}{3}(\overline{w_iw_j}\cdot\overline{w_kw_l}+\overline{w_iw_k}\cdot\overline{w_jw_l}+\overline{w_iw_l}\cdot\overline{w_jw_k})$,

(b) $\overline{p_{,i}'w_jw_k}=-c_{ijk}+k_{(jk)}\overline{p_{,i}\cdot w_jw_k}$ (not summed).

(a) is satisfied by a simple harmonic fluctuation; (b) is a quite arbitrary assumption, the constants being left for determination by experiment.

The theory is applied to the case of pressure flow between parallel planes. With some simplifying hypotheses and the assumption of a very large Reynolds number, the equation

$$y\frac{dU}{dy}+\gamma^2\frac{d^2}{dy^2}\left(y\Big/\frac{dU}{dy}\right)=0, \qquad \gamma=\text{constant},$$

is obtained for the mean velocity U, y being measured across the channel $(-d\leqq y\leqq d)$. Introduction of an auxiliary variable ϕ gives

$$dU/d\sigma=-\gamma\sigma/\phi, \quad d^2\phi/d\sigma^2=-\sigma^2/\phi, \qquad \sigma=y/d.$$

The second of these equations is integrated numerically with the boundary conditions $d\phi/d\sigma=0$ for $\sigma=0$, $\phi=0$ for $\sigma=1$. Then U is found from the first equation to within a constant of integration. For comparison with experimental results, this constant is given a suitable value.

J. L. Synge (Toronto, Ont.).

Küssner, H. G. Das zweidimensionale Problem der beliebig bewegten Tragfläche unter Berücksichtigung von Partialbewegungen der Flüssigkeit. Luftfahrtforschung **17**, 355–361 (1940). [MF 6563]

Schwarz, L. Berechnung der Funktionen $U_1(s)$ und $U_2(s)$ für grössere Werte von s. Luftfahrtforschung **17**, 362–369 (1940). [MF 6564]

By means of a Laplace transformation the two-dimensional linearized problem of the flow around a wing in arbitrary motion can be reduced to that of the flow around a harmonically oscillating wing. The pressure distribution of a wing in arbitrary motion is represented by an integral and the functions occurring therein are tabulated.

W. Prager (Providence, R. I.).

Küssner, H. G. und Schwarz, L. Der schwingende Flügel mit aerodynamisch ausgeglichenem Ruder. Luftfahrtforschung **17**, 337–354 (1940). [MF 6600]

The authors study the two-dimensional flow around an airfoil with an aerodynamically balanced elevator whose trailing edge is fitted with a tab. Fin, elevator and tab are replaced by a straight line each. Formulae for the pressure distribution are derived and the functions occurring therein are tabulated. *W. Prager* (Providence, R. I.).

Schade, Th. Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage. I. Analytischer Teil. Luftfahrtforschung **17**, 387–400 (1940). [MF 6566]

Using L. Prandtl's method of the acceleration potential, the author studies the pressure distribution of a thin vibrating airfoil of circular plan-form. The (small) amplitudes of the transverse vibrations are assumed to be such that at any instant the airfoil forms part of a quadric, all points oscillating with the same frequency, but not necessarily with the same phase. *W. Prager* (Providence, R. I.).

Dietze, F. Zum Luftkraftgesetz der harmonisch schwingenden, knickbaren Platte (Flügel mit Ruder und Hilfsruder). Luftfahrtforschung **18**, 135–141 (1941). [MF 6798]

In an earlier paper [Luftfahrtforschung **16**, 84–96 (1939)] the author obtained formulas for the forces and moments on a two-dimensional thin airfoil periodically deformed. This included the case of a bent-plate airfoil having an oscillating aileron and aileron tab. The formulas were obtained by application of the theory of airfoils in nonuniform motion of Küssner [Luftfahrtforschung **13**, 410–424 (1936)]. The present paper constitutes an extension of the earlier calculation to the case wherein the aileron and/or tab are equipped with aerodynamic balance. This extension is made in an approximate manner, since the effects of slots between the control elements and of projecting noses are neglected. The resulting formulas, which are lengthy, are presented in

a tabular form suitable for application to problems of wing and control-surface flutter. *W. R. Sears.*

Schwarz, L. Berechnung der Druckverteilung einer harmonisch sich verformenden Tragfläche in ebener Strömung. Luftfahrtforschung 17, 379–386 (1940). [MF 6565]

The author investigates the two-dimensional flow around a thin airfoil executing harmonic oscillations which change the shape as well as the position of the airfoil. Pressure distribution, lift and moment are represented by integrals which for vanishing frequency reduce to Munk's integrals. *W. Prager* (Providence, R. I.).

Bilharz, Herbert. Rollstabilität eines um seine Längsachse freien Flugzeugs bei automatisch gesteuerten, intermittierenden, konstanten Querrudermomenten. Luftfahrtforschung 18, 317–326 (1941). [MF 6702]

The mathematical problem amounts to the determination of the conditions under which the ordinary differential equation

$$\varphi''+a\varphi'+b\ \mathrm{sgn}\ (\varphi+c\varphi')=0$$

($a>0$, $b>0$ and c are constants) possesses "stable" solutions. A solution $\varphi(t)$ is called stable if it has denumerably many zeros t_ν and if there exist two positive numbers p and q such that $p\leqq t_{\nu+1}-t_\nu\leqq q$, uniformly for all ν. The problem is discussed in the phase-plane φ, φ'; stable solutions are found to exist for negative values of c. From the point of view of the theory of differential equations this result does not seem to be new. *W. Prager.*

Proudman, J. On Laplace's differential equations for the tides. Proc. Roy. Soc. London. Ser. A. 179, 261–288 (1942). [MF 6237]

The present paper is an investigation of the theory of forced oscillations of a homogeneous ocean on the rotating earth. Results derived from Solberg's equations [Astrophys. Norvegica 1, 237–340 (1936)] are compared with the less elaborate Laplace's theory. According to the author, his chief results lead to a considerable modification of the impression created by Norwegian scientists. Let $P=p/\rho+V$, where p denotes the pressure, ρ the density and V the potential of the gravitational and centrifugal forces. Using Solberg's equations of motion and introducing cylindrical coordinates $\tilde{\omega}$, χ, z the author obtains for P the differential equation

$$(*)\qquad \frac{1}{\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}\left(\tilde{\omega}\frac{\partial P}{\partial\tilde{\omega}}\right)+\frac{1}{\tilde{\omega}^2}\frac{\partial^2 P}{\partial\chi^2}+\left(1-\frac{4\omega^2}{\sigma^2}\right)\frac{\partial^2 P}{\partial z^2}=0,$$

where ω and $2\pi/\sigma$ denote the angular speed of the earth and the period of the oscillation, respectively. All relevant quantities: the components of the velocity and the boundary conditions can be expressed in terms of P. The author gives the particular solutions of (*) corresponding to a narrow ocean bounded by meridians, a canal along a parallel of latitude, a flat circular sea near to the North Pole, a broad equatorial channel and so on.

The equation (*) is similar to that considered by M. Brillouin and J. Coulomb [Gauthiers-Villers, Paris, 1933]. In this connection it should be mentioned that according to J. Hadamard [Enseignement Math. 35, 5–42 (1936), especially p. 39] the following question still requires elucidation: the equation of Brillouin-Coulomb, as well as the equation (*), is elliptic when $\sigma^2>4\omega^2$, and hyperbolic when $\sigma^2<4\omega^2$, whereas in both cases the boundary conditions are of the type suitable for the elliptic case only. *A. Weinstein.*

Chapman, S. and Cowling, T. G. The velocity of diffusion in a mixed gas; the second approximation. Proc. Roy. Soc. London. Ser. A. 179, 159–169 (1941). [MF 5736]

The authors have developed the theory of nonuniform gases [The Mathematical Theory of Non-uniform Gases, Cambridge University Press, Cambridge, England, 1939; these Rev. 1, 187], where successive approximations to the velocity-distribution function f are sought. The function f is approximated by means of functions $f^{(0)}, f^{(1)}, f^{(2)}, \cdots$, where $f^{(0)}$ is Maxwell's velocity-distribution function for a uniform gas, and $f^{(1)}$, in the case of a single gas, has two components, one proportional to certain combinations of the gradients of the velocity components, the other proportional to the temperature gradient. In a mixed gas, $f^{(1)}$ contains also terms which are proportional to the gradients of concentration and pressure, as well as to the external forces acting on the molecules; these terms modify the equations of thermal conduction and diffusion. The $f^{(2)}$, which is much more complicated than $f^{(1)}$, was completely determined by Burnett [Proc. London Math. Soc. (2) 40, 382–435 (1935)]. In this paper the authors investigate the modifications introduced in the ordinary equation of diffusion for the case of a mixed gas when $f^{(2)}$ is taken into account. They use the approximative method described in their book for the case of a simple gas. These corrections add nine new terms to the expression for the velocity of diffusion; and each term involves as a factor a new second order diffusion coefficient. All these new terms vanish if the gas is uniform. The expressions for the nine new diffusion coefficients are extremely complicated, and have been evaluated only approximately. *F. Cernuschi* (Tucumán).

Van Mieghem, J. Quelques remarques sur de nouvelles équations approchées du mouvement horizontal de l'air. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 27, 339–345 (1941). [MF 6892]

A further discussion of the author's approximate equation for the horizontal components of the wind [Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. 19, fasc. 3, 65 pp. (1941); these Rev. 3, 219]. Expressions for the direction and acceleration of the wind, the curvature of the streamlines, etc., are derived and discussed. *H. W. Liepmann.*

Fubini, Guido. An elementary observation on the equations of external ballistics. Math. Notae 2, 3–10 (1942). (Spanish) [MF 6738]

F. R. Moulton [New Methods in Exterior Ballistics, University of Chicago Press, Chicago, Ill., 1926] obtained the explicit differential equations for the motion of a spinning projectile about its center of gravity. The derivation is limited to the case where the trajectory of the center of gravity is a plane curve, or may be considered so as a first approximation. In the paper under review the author makes some simple observations by means of which the equations for the most general case can be derived from those valid when, as a first approximation, the trajectory is regarded as a straight line. The angles of Euler are taken as θ (nutation), ψ (precession) and φ (rotation about axis of shell). Expressions are obtained for θ', ψ' and φ' in terms of θ, ψ, φ and the components of angular velocity of a suitably chosen coordinate system. The coordinate system and angular velocity may be modified to fit various hypotheses.

W. E. Milne (Corvallis, Ore.).

Boulton, N. S. The steady flow of ground-water to a pumped well in the vicinity of a river. Philos. Mag. (7) **33**, 34–50 (1942). [MF 6227]

The treatment of the problem indicated in the title is restricted by a number of rather arbitrary conditions. The crossection of the riverbed is considered to be semicircular, and filled to the top; the longitudinal gradient of the river, which causes the flow, is neglected. The fact that the author replaces the real boundary conditions at the well by an assumed "line sink" of constant intensity along the well axis is more important. The solution of the problem thus defined is given in the form of an infinite series of modified Bessel functions of zero order. Its discussion shows that, under the assumptions made and other conditions remaining the same, an increase in the distance separating the well from the river increases the extension of the cone of depression in the direction away from the river, but has little or no effect in the other direction toward the river.

P. Neményi (Fort Collins, Colo.).

Rossbach, H. F. Über Grundwasserströmungen mit freier Oberfläche. Ing.-Arch. **12**, 221–246 (1941). [MF 6424]

Three simplified and idealized cases of drainage are treated: (1) drainage by a straight open ditch with a rectangular cross-section, assuming that the groundwater-bearing stratum is of infinite thickness; (2) same problem, with the sides of the ditch assumed impermeable; (3) thickness of the water-bearing stratum is limited by an underlying impermeable layer, upon which a straight drainage pipe is laid, where this pipe is idealized by a "line sink." Obviously all three problems can be treated as plane problems of potential flow. The author uses the known hodograph methods and the Schwarz-Christoffel theorem.

P. Neményi (Fort Collins, Colo.).

Elasticity

Garcia, Godofredo. On the parametric representation of elastic (seismic) waves in anisotropic media in a hexagonal crystalline system (with transversal isotropy, internal waves). Actas Acad. Ci. Lima **4**, 125–144 (1941). (1 plate) (Spanish) [MF 6028]

Garcia, Godofredo. Parametric representation of elastic (seismic) waves in anisotropic media in a hexagonal crystalline system (form of the surface waves of Lord Rayleigh which diminish with depth). Revista Ci., Lima **43**, 545–566 (1941)=Actas Acad. Ci. Lima **4**, 191–212 (1941). (Spanish) [MF 6224]

Goodier, J. N. An extension of Saint-Venant's principle, with applications. J. Appl. Phys. **13**, 167–171 (1942).

Simple energy considerations, which have previously been employed [J. N. Goodier, Philos. Mag. (7) **23**, 607–609; (7) **24**, 325 (1937) and O. Zanaboni, Atti Accad. Naz. Lincei. Rend. **25**, 117–121, 595–601 (1937)] to provide a rational basis for the principle of Saint-Venant, are shown to lead to the conclusion that forces applied in the neighborhood of a rigidly fixed portion of an elastic solid can cause only local stress and strain. This principle indicates useful features of certain problems encountered in engineering, and is used here in the calculation of the stiffness of a thin layer of elastic material confined between rigid plates, under any kind of load applied to the plates.

Author's summary.

Sen, Bibhutibhusan. Note on the bending of thin uniformly loaded plates bounded by cardioids, lemniscates, and certain other quartic curves. Philos. Mag. (7) **33**, 294–302 (1942). [MF 6592]

Using various systems of isometric parameters, the author determines the displacements of clamped plates of the types described in the title. The results are particularly interesting due to the fact that they consist of a finite number of terms.

W. Prager (Providence, R. I.).

Sen, Bibhutibhusan. Stresses due to forces and couples acting in the interior of a semi-infinite elastic solid. Bull. Calcutta Math. Soc. **32**, 73–83 (1940). [MF 6159]

The author determines the stresses and displacements produced in a semi-infinite elastic solid by the following loads applied to an interior point of the solid: (1) a single force whose line of action is perpendicular to, and (2) a couple whose plane of action is parallel to the plane surface of the solid, (3) a center of compression. In all three cases the surface of the solid is supposed to be free from stress.

W. Prager (Providence, R. I.).

Seth, B. R. Finite strain in a rotating shaft. Proc. Indian Acad. Sci., Sect. A. **14**, 648–651 (1941). [MF 6478]

The paper is concerned with the determination of stresses and displacements in a rotating circular shaft. The relations between strains and displacement-derivatives are assumed to include terms of the second order, but the stress-strain relations are taken in the form of the usual Hooke's law. The shaft is assumed to be in the state of plane strain. A particular solution (for a certain angular velocity of the shaft) is obtained and the resulting expressions for stresses are similar to those of the linear theory. Specific calculations indicate that the corresponding stresses in the linearized problem may differ by 10 to 12 per cent from those obtained by the author. *I. S. Sokolnikoff* (Madison, Wis.).

Platrier, Charles. Milieux homogènes isotropes en rotation uniforme autour d'un axe et dans lesquels il n'existe pas de tension sur des sections orthogonales à cet axe. Problème de la meule. C. R. Acad. Sci. Paris **211**, 221–224 (1940). [MF 5347]

The author gives a solution of the equations of (relative) equilibrium and of the Beltrami conditions of compatibility in the case that the external forces reduce to the centrifugal force. *A. Weinstein* (Toronto, Ont.).

Hay, G. E. The finite displacement of thin rods. Trans. Amer. Math. Soc. **51**, 65–102 (1942). [MF 6089]

This paper contains a systematic treatment of the problem of finite displacement of a curved isotropic rod of uniform cross-section, under the action of external forces applied at the ends of the rod. The strains are assumed to be small, but the stress-strain relations are taken to contain five elastic constants instead of the usual two. The equations of equilibrium, the equations of compatibility, the boundary conditions and the macroscopic equations of equilibrium (involving an integration over the cross-section of the rod) are derived by tensor methods. These equations are solved by writing the expansions for the components of the stress and strain tensors in power series of a small parameter $\epsilon = D/l$, where D is the average diameter of the unstrained cross-section and l is the length of the unstrained rod, and by deducing the equations for the coefficients of the powers of ϵ in these expansions. The solution of the latter equations yields a theoretical solution of the problem

to any desired degree of accuracy. The possibility of an experimental determination of the values of the new elastic constants appears from a consideration of the problem of straightening of an initially curved rod, where one functional relation involving two new elastic constants arises. An appendix to the paper contains a proof of Saint Venant's hypothesis for a thin rod subjected to a finite displacement. *I. S. Sokolnikoff* (Madison, Wis.).

Klinger, Friedrich. Der elastische Schwerpunkt am räumlichen Bogenträger. Akad. Wiss. Wien, S.-B. IIa. **149**, 269–289 (1940). [MF 4939]

The author reproduces the equations of the theory of curved elastic rods in the language of the calculus of dyades. He follows the notations of M. Lagally's "Vorlesungen über Vectorrechnung" [Leipzig, 1928], presupposing familiarity with the contents of this text. *E. Reissner.*

Hrennikoff, A. Solution of problems of elasticity by the framework method. J. Appl. Mech. **8**, A-169–A-175 (1941). [MF 6232]

This paper generalizes a procedure of K. Wieghardt [Verhand. des Vereins zur Beforderung des Gewerbefleisses **85**, 137–176 (1906)]. A stressed isotropic medium is replaced by a framework of bars arranged in a pattern compatible with the conditions of isotropic deformability and endowed with relevant elastic properties. The resultant structure is assumed subject to the prescribed loading conditions and is then analyzed by a method of successive approximations analogous to the Hardy-Cross moment-distribution procedure. The solution of problems in the theory of plane stress is illustrated by an explicit example, while an indication is made of the modifications necessary for the analysis of plates and shells and for the general treatment of three-dimensional problems. *F. B. Hildebrand.*

Payne, J. H. Torsion in box beams. Aircraft Engrg. **14**, 2–8 (1942). [MF 6360]

The author studies the effect of axial constraint on the stresses in a box beam consisting of two spars and two panels, which is subject to torsion. The torque is resolved into two couples, the forces of which are considered as loads bending the spars and panels, and the elementary theory of bending is applied. Various cases of torque loading are studied and a table of the corresponding bending moments of the spar is given. *W. Prager* (Providence, R. I.).

Higgins, Thomas James. The torsion of a prism with cross section the inverse of an ellipse. J. Appl. Phys. **13**, 457–459 (1942).

Saint-Venant's torsion problem for a right prism whose cross section boundary is a curve the inverse of an ellipse with respect to its center is solved by the method of Rosa M. Morris [Math. Ann. **116**, 374–400 (1939)]. Explicit results, all finite in form, are given for the torsion function, the twisting moment and the shearing stresses. The special case of a right circular prism is deducible from the results. The problem is of interest largely because it is another example of the very few torsion problems for which the results are expressible in finite form. *D. L. Holl* (Ames, Iowa).

Friedrichs, K. O. and Stoker, J. J. Buckling of the circular plate beyond the critical thrust. J. Appl. Mech. **9**, A-7–A-14 (1942). [MF 6265]

A flat plate subjected to compressive forces at its edges and in its plane becomes instable when the compressive forces reach certain critical values, but the forces can be increased considerably without causing a collapse of the plate. The reason for this behavior is that when a plate, the edge of which is restrained, buckles it experiences at the same time a stretching of its surface. This stretch relieves the central part of the plate of a portion of the compressive stresses and renders the plate relatively stiffer than a column. The mathematical formulation of this buckling problem was given by von Kármán and was solved for a circular supported plate by the authors [Amer. J. Math. **63**, 839–888 (1941); these Rev. **3**, 223] under the assumption of radial symmetry. In the present paper the authors give a detailed numerical solution for all values of the ratio $\Lambda = p/p_E$, where p is the prescribed edge thrust and p_E the critical value of the linearized buckling problem. The problem, which is a nonlinear one, is solved by two methods for finite values of Λ: the perturbation and the power series methods. The latter can be used for a much higher range of values of Λ than the former. The number of coefficients necessary for sufficient accuracy increases with Λ. For $\Lambda = 14.7$, 30 terms are needed. The most notable single result is that the membrane stresses for large values of Λ become tensions in the interior of the plate and change abruptly to compressions in a narrow boundary layer at the edge of the plate. Numerous curves showing the behavior of the deflection and the stresses are given for a series of values of Λ. The limit state, which is discussed at the end of the paper, is approached for relatively small values of Λ, that is, $\Lambda > 5$. The paper gives not only the numerical results but also the mathematical theory to the extent needed by an applied mathematician or an engineer and can be read independently of previous publications. *A. Weinstein* (Toronto, Ont.).

Stoker, J. J. Mathematical problems connected with the bending and buckling of elastic plates. Bull. Amer. Math. Soc. **48**, 247–261 (1942). [MF 6403]

The purpose of this address delivered before the American Mathematical Society is to stimulate interest in applied mathematics not only for the furtherance of practical ends but also for the advancement of mathematics as a whole. The author gives a short but interesting review of the development of the theory of plates since the times of James Bernoulli the Younger, Lagrange, Sophie Germain, Navier and Cauchy. The main part of the paper is devoted to the discussion of the recent papers by K. O. Friedrichs and J. J. Stoker on the nonlinear buckling problem for a circular plate [see the preceding review]. The most striking feature is that this buckling problem appears as a "boundary layer problem": with increasing compressive load all quantities tend to become constant in an increasingly large portion of the interior of the plate and to change rapidly in a narrow strip, or layer, near its edge. At the end of his address the author formulates several unsolved problems of the theory of plates. *A. Weinstein* (Toronto, Ont.).

Woinowsky-Krieger, S. Über die Eigenschwingungen eines Kreisbogenträgers mit drei Gelenken. Ing.-Arch. **12**, 1–5 (1941). [MF 6859]

The author investigates the free vibrations of a circular three joint structure under uniform radial pressure. It is remarked that small deviations from a circular form influence the results only slightly. The analysis assumes a constant length and a constant cross section of the arches and the calculation of the characteristic frequencies is carried out by the Rayleigh-Ritz method. *A. E. Heins.*